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Digital Servo Loop Gain Optimization for Second and Third Order Systems

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ABSTRACT

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In this paper, an optimum loop-gain-frequency function is derived for a continuous linear servo system of second or third order operating with incremental data.

A ramp system input is closely approximated by integrating a pulse sequence and using the resulting sequential step function. The difference between the ramp and the system output is taken as error. The error squared integral is used as a criterion of the goodness of approximation of the ramp by the system output and this integral is minimized for second and third order linear systems. Digital computer techniques are used to minimize this integral.

In both the second and third order systems, the functional relationship between the loop gain and frequency which minimizes the error squared integral is found to be linear. Furthermore, the family of linear minimizing functions generated in the third order case is of the same slope as the second order minimizing function. It is also shown that by choosing the loop gain and frequency correctly, it is possible to obtain values of the error

squared integral very close to zero provided the pulse sequence is not delayed more than a small fraction of the pulse period.

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LIST OF SYMBOLS

1.	a	Constants
2.	b	
3.	β	
4.	$e(t)$	Error
5.	g	Constants
6.	γ	
7.	$G(s)$	Laplace Transforms of System Elements
8.	$H(s)$	
9.	I	Error Squared Integral
10.	Im	Imaginary Part
11.	j	$\sqrt{-1}$
12.	k	Summation Variable
13.	K	Upper Summation Limit
14.	$r(t)$	System Input
15.	$R(s)$	Laplace Transform of System Input
16.	Re	Real Part
17.	S	Laplace Variable
18.	t	Time

19.	T	Pulse Period
20.	$\theta(t)$	System Output
21.	$\theta(s)$	Laplace transform of System Output
22.	x	Time Variable (Zero at time KT)
23.	$y(t)$	Desired Output
24.	Z	Variable Pulse Period αT
25.	ζ	Damping Ratio

The following variables which can be defined in terms of those above were used to make notations more compact.

26. $A = (1 - \zeta^2)^{1/2}/\zeta$

27. $\beta_0 = (\alpha^2 + \beta^2)^{1/2}$

28. $D = (M - jN) / (e^{-(\alpha - j\beta)T} - 1)$

29. $G = (1 - \lambda)^2 + A^2$

30. $H = \lambda - 1 + A^2$

31. $\lambda = \gamma T/Z$

32. $M = \alpha(\gamma - \alpha) + \beta^2$

33. $N = 2\alpha\beta - \beta\gamma$

34. $P = 2 - \lambda$

35. $\phi = \tan^{-1} \beta/-\alpha$

36. $\phi_1 = \tan^{-1}(\beta/-\alpha) + \tan^{-1}(\beta/\gamma - \alpha)$

INTRODUCTION

The purpose of the following analysis is to derive an optimum functional relationship between the d. c. loop gain and pulse frequency for a linear continuous servo system using incremental data. Second and third order systems operating with a sequential step input will be analyzed.

A block diagram for a general linear feedback system is shown in Figure 1. The transfer function for this system is:

$$\frac{\theta(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)} \quad . \tag{1-1}$$

In the following analysis -

I. System output $\theta(t)$ is required to describe, as closely as possible, a linear function of time. The desired output is:

$$Y(t) = ct. \tag{1-2}$$

II. The reference input $r(t)$ is a sequential step function (integrated pulse train). See Figure 2.

$$r(t) = \sum_{0}^{k} \mu(t - kT) \tag{1-3}$$

III. Actual output $\theta(t)$ will also consist of a series of steps. However, their shape will be determined by physical characteristics of a particular system.

IV. The first unit step is applied with the system at rest; but, in general, succeeding steps are applied with variable initial conditions which depend upon the value and slope of $\theta(t)$ at the instant the step is applied. See Figure 3.

V. The difference between the actual output $\theta(t)$ and the desired output $Y(t)$ will be called error.

$$e(t) = \theta(t) - Y(t) \tag{1-4}$$

As a criterion of the goodness of approximation of the desired output by the actual output, the error squared integral

$$I = \frac{1}{T}\int_{KT}^{(K+1)T} [e(t)]^2 \, dt \tag{1-5}$$

is used. K is taken large enough to assure steady-state behavior; steady-state in the sense that initial conditions will not vary essentially from pulse K to pulse K + 1. Reasons for this choice of criterion are discussed in Sokolnikoff; "Advanced Calculus", page 378.

In order to simplify the analysis, the first unit step input occurs at $t = 0$. Transfer functions have also been selected in a form that will simplify equations.

VI. The problem is to find a loop-gain-frequency relationship which minimizes I.

DERIVATION OF THE ERROR SQUARED INTEGRAL FOR THE SECOND ORDER SYSTEM

The transfer function for a second order underdamped system can be written in the form:

$$\frac{\theta(s)}{R(s)} = \frac{\beta_0^2}{(s+\alpha)^2+\beta^2} \tag{2-1}$$

where α and β are constants and $\beta_0^2 = \alpha^2 + \beta^2$. R(s) can be found by transforming equation 1-3 into the domain of the Laplace variable s.

$$R(s) = \frac{1}{s}\left[1 + e^{-st} + e^{-st} + \ldots\ldots\right] \tag{2-2}$$

Substituting equation 2-2 into equation 2-1:

$$\theta(s) = \frac{\beta_0^2}{(s+\alpha)^2+\beta^2} \cdot \frac{1}{s}\left[1 + e^{-st} + e^{-2st} + \ldots\ldots\right] \tag{2-3}$$

Transforming equation 2-3 into the time domain:

$$\theta(t) = 1 + \frac{\beta_0}{\beta} e^{-\alpha t} \sin[\beta t - \phi]$$

$$+ 1 + \frac{\beta_0}{\beta} e^{-\alpha(t-T)} \sin[\beta(t-T) - \phi]$$

$$+ -$$

$$- -$$

$$- -$$

$$+ 1 + \frac{\beta_0}{\beta} e^{-\alpha(t-KT)} \sin[\beta(t-KT) - \phi]$$

$$\theta(t) = K + 1 + \frac{\beta_0}{\beta} \sum_{k=0}^{K} e^{-\alpha(t-kT)} \sin[\beta(t-kT) - \phi], \tag{2-4}$$

where $\phi = \tan^{-1} \frac{\beta}{-\alpha}$.

Changing the sine function to exponential form,

$$e^{-\alpha(t-kT)} \sin[\beta(t-kT) - \phi]$$

$$= \frac{e^{-\alpha t} \cdot e^{\alpha kT}\left[e^{j[\beta(t-kT) - \phi]} - e^{-j[\beta(t-kT) - \phi]}\right]}{2j}$$

$$= \frac{e^{-\alpha t} \cdot e^{j[\beta t - \phi]} \cdot e^{\alpha kT} \cdot e^{-j\beta kT}}{2j}$$

$$\frac{-e^{-\alpha t} \cdot e^{j[\beta t - \phi]} \cdot e^{\alpha kT} \cdot e^{j\beta T}}{2j} \quad . \tag{2-5}$$

A summation of terms involving k from equation 2-5 can be expressed in closed form since they form a geometric series.

$$\sum_{k=0}^{K} e^{kT(\alpha + j\beta)} = \frac{1 - e^{(K+1)(\alpha + j\beta)T}}{1 - e^{(\alpha + j\beta)T}} \tag{2-6}$$

$$\sum_{k=0}^{K} e^{kT(\alpha - j\beta)} = \frac{1 - e^{(K+1)(\alpha - j\beta)T}}{1 - e^{(\alpha - j\beta)T}} \tag{2-7}$$

At this point let a new time variable x be assigned with zero reference at t = KT. Then

$$t = KT + x. \tag{2-8}$$

Making use of equations 2-6, 2-7, and 2-8, the summation of equation 2-4 becomes:

$$\sum_{k=0}^{K} e^{-\alpha(t-kT)} \sin[\beta(t-kT) - \phi]$$

$$= \frac{1}{2j} \left[e^{-\alpha(KT+x)} \cdot e^{j[\beta(KT+x) - \phi]} \frac{1 - e^{(K+1)(\alpha - j\beta)T}}{1 - e^{(\alpha - j\beta)T}} \right.$$

$$\left. - e^{-\alpha(KT+x)} \cdot e^{-j[\beta(KT+x) - \phi]} \frac{1 - e^{(K+1)(\alpha + j\beta)T}}{1 - e^{(\alpha + j\beta)T}} \right]$$

$$= \frac{1}{2j} \left[e^{-KT(\alpha - j\beta)} \cdot e^{-x(\alpha - j\beta)} \cdot e^{-j\phi} \frac{1 - e^{(K+1)(\alpha - j\beta)T}}{1 - e^{(\alpha - j\beta)T}} \right.$$

$$\left. - e^{-KT(\alpha + j\beta)} \cdot e^{-x(\alpha + j\beta)} \cdot e^{j\phi} \frac{1 - e^{(K+1)(\alpha + j\beta)T}}{1 - e^{(\alpha + j\beta)T}} \right]$$

$$\tag{2-9}$$

Expressing ϕ in terms of α and β,

$$e^{j\phi} = \cos\left[\tan^{-1}\frac{\beta}{-\alpha}\right] + j\sin\left[\tan^{-1}\frac{\beta}{-\alpha}\right]$$

$$= \frac{-\alpha}{\beta_0} + j\frac{\beta}{\beta_0} = -\frac{1}{\beta_0}\left[\alpha - j\beta\right] \quad . \tag{2-10}$$

It follows that

$$e^{-j\phi} = -\frac{1}{\beta_0}\left[\alpha + j\beta\right] . \tag{2-11}$$

Using equations 2-10 and 2-11, equation 2-9 becomes

$$\sum_{k=0}^{K} e^{-\alpha(t-kT)} \sin\left[\beta(t-kT) - \phi\right]$$

$$= -\frac{\alpha + j\beta}{2j\beta_0} e^{-(\alpha - j\beta)x} \frac{e^{-KT(\alpha - j\beta)} - e^{(\alpha - j\beta)T}}{1 - e^{(\alpha - j\beta)T}}$$

$$+ \frac{\alpha - j\beta}{2j\beta_0} e^{-(\alpha + j\beta)x} \frac{e^{-KT(\alpha + j\beta)} - e^{(\alpha + j\beta)T}}{1 - e^{(\alpha + j\beta)T}} \tag{2-12}$$

Substituting equation 2-12 into equation 2-4 and taking K large to assure steady state response, the output becomes

$$\theta(x) = K + 1 + \frac{\alpha + j\beta}{2j\beta} \cdot \frac{e^{-(\alpha - j\beta)x}}{e^{-(\alpha - j\beta)T} - 1}$$

$$+ \frac{\alpha - j\beta}{2j\beta} \cdot \frac{-e^{-(\alpha + j\beta)x}}{e^{-(\alpha + j\beta)T} - 1}$$

(2-13)

The slope of the desired output function y(t) is $\frac{1}{T}$. From equations 1-2 and 2-8, it follows that

$$Y(x) = K + \frac{x}{T} \tag{2-14}$$

Subtracting equation 2-14 from equation 2-13 and squaring the result, the error squared integral for the second order system becomes:

$$I = \frac{1}{T} \int_0^T \Bigg\{ 1 + \frac{x^2}{T^2} - \frac{2x}{T} + \frac{(\alpha + j\beta)^2}{-4\beta^2} \frac{e^{-2(\alpha - j\beta)x}}{(e^{-(\alpha - j\beta)T} - 1)^2}$$

$$+ \frac{(\alpha - j\beta)^2}{-4\beta^2} \frac{e^{-2(\alpha + j\beta)x}}{(e^{-(\alpha + j\beta)T} - 1)^2} + \frac{\beta_0^2}{2\beta^2} \frac{e^{-2\alpha x}}{(e^{-(\alpha - j\beta)T} - 1)(e^{-(\alpha + j\beta)T} - 1)}$$

$$+ \left[1 - \frac{x}{T}\right] \left[\left(\frac{(\alpha + j\beta)(e^{-(\alpha - j\beta)x})}{j\beta(e^{-(\alpha - j\beta)T} - 1)} \right) - \left(\frac{(\alpha - j\beta)\, e^{-(\alpha + j\beta)x}}{j\beta(e^{-(\alpha + j\beta)T} - 1)} \right) \right] \Bigg\} dx$$

(2-15)

$$\int_0^T e^{-(\alpha - j\beta)x} dx = - \frac{1}{(\alpha - j\beta)} \left(e^{-(\alpha - j\beta)T} - 1\right)$$

(2-16)

$$\int_0^T x\, e^{-(\alpha - j\beta)x}\, dx = -\frac{x}{\alpha - j\beta}\, e^{-(\alpha - j\beta)x} \Bigg|_0^T$$

$$+ \frac{1}{(\alpha - j\beta)} \int_0^T e^{-(\alpha - j\beta)x}\, dx$$

$$= -\frac{T}{\alpha - j\beta}\, e^{-(\alpha - j\beta)T} - \frac{1}{(\alpha - j\beta)^2}\left(e^{-(\alpha - j\beta)T} - 1\right)$$

(2-17)

Integrating equation 2-15 with the aid of equations 2-16 and 2-17:

$$I = 1 + \frac{T^3}{3T^3} - \frac{2T^2}{2T^2} + \frac{(\alpha + j\beta)}{j\beta T(e^{-(\alpha - j\beta)T} - 1)} \cdot \frac{(1 - e^{-(\alpha - j\beta)T})}{(\alpha - j\beta)}$$

$$- \frac{(\alpha - j\beta)}{j\beta T(e^{-(\alpha + j\beta)T} - 1)} \cdot \frac{(1 - e^{-(\alpha + j\beta)T})}{\alpha + j\beta}$$

$$+ \frac{(\alpha + j\beta)\ (-1)}{j\beta T^2\, (e^{-(\alpha - j\beta)T} - 1)} \left[- \frac{Te^{-(\alpha - j\beta)T}}{\alpha - j\beta} - \frac{(e^{-(\alpha - j\beta)T} - 1)}{(\alpha - j\beta)^2} \right]$$

$$+ \frac{(\alpha - j\beta)}{j\beta T^2\, (e^{-(\alpha + j\beta)T} - 1)} \left[- \frac{Te^{-(\alpha + j\beta)T}}{\alpha + j\beta} - \frac{(e^{-(\alpha + j\beta)T} - 1)}{(\alpha + j\beta)^2} \right]$$

$$+ \frac{(\alpha + j\beta)^2\, (e^{-2(\alpha - j\beta)T} - 1)}{8\beta^2 T(\alpha - j\beta)\, (e^{-(\alpha - j\beta)T} - 1)^2} + \frac{(\alpha - j\beta)^2\, (e^{-2(\alpha + j\beta)T} - 1)}{8\beta^2 T(\alpha + j\beta)\, (e^{-(\alpha + j\beta)T} - 1)^2}$$

$$+ \frac{\beta_0^{\;2}\,(e^{-2\alpha T}-1)}{-4\alpha\beta^2 T(e^{-(\alpha - j\beta)T}-1)\,(e^{-(\alpha+j\beta)T}-1)}$$

(2-18)

Simplifying,

$$I = \frac{1}{3} - \frac{\alpha + j\beta}{j\beta T(\alpha - j\beta)} + \frac{\alpha - j\beta}{j\beta T(\alpha + j\beta)} + \frac{\alpha + j\beta}{j\beta T^2(\alpha - j\beta)^2} - \frac{\alpha - j\beta}{j\beta T^2(\alpha + j\beta)^2}$$

$$+ \frac{(\alpha + j\beta)\,(e^{-(\alpha - j\beta)T})}{j\beta T(\alpha - j\beta)\,(e^{-(\alpha - j\beta)T}-1)} + \frac{(\alpha - j\beta)\,(-e^{-(\alpha + j\beta)T})}{j\beta T(\alpha + j\beta)\,(e^{-(\alpha + j\beta)T}-1)}$$

$$+ \frac{(\alpha + j\beta)^2\,(e^{-2(\alpha - j\beta)T}-1)}{8\beta^2 T(\alpha - j\beta)\,(e^{-(\alpha - j\beta)T}-1)^2} + \frac{(\alpha - j\beta)^2\,(e^{-2(\alpha + j\beta)T}-1)}{8\beta^2 T(\alpha + j\beta)\,(e^{-(\alpha + j\beta)T}-1)^2}$$

$$- \frac{\beta_0^{\;2}\,(e^{-2\alpha T}-1)}{4\alpha\beta^2 T(e^{-(\alpha - j\beta)T}-1)\,(e^{-(\alpha+j\beta)T}-1)}$$

(2-19)

Equation 2-19 can be simplified by combining terms which appear in conjugate pairs.

$$2\mathrm{Re}\left[\frac{\alpha + j\beta}{-j\beta T(\alpha - j\beta)}\right] = 2\mathrm{Re}\left[\frac{(\alpha + j\beta)^2}{-j\beta T\beta_0^{\;2}}\right] = \frac{4\alpha\beta j}{-j\beta T\beta_0^{\;2}} = -\frac{4\alpha}{\beta_0^{\;2}}$$

(2-20)

$$2\mathrm{Re}\left[\frac{\alpha + j\beta}{j\beta T^2(\alpha - j\beta)^2}\right] = 2\mathrm{Re}\left[\frac{(\alpha + j\beta)^3}{j\beta T^2 \beta_0{}^4}\right]$$

$$= 2\left[\frac{3\alpha^2\beta - \beta^3}{\beta T^2 \beta_0{}^4}\right] = \frac{6\alpha^2 - 2\beta^2}{\beta_0{}^4 T^2}$$

(2-21)

$$2\mathrm{Re}\left[\frac{(\alpha + j\beta)\,(e^{-(\alpha - j\beta)T})}{j\beta T(\alpha - j\beta)\,(e^{-(\alpha - j\beta)T} - 1)}\right] = 2\mathrm{Re}\left[\frac{(\alpha + j\beta)^2}{j\beta T \beta_0{}^2(1 - e^{(\alpha - j\beta)T})}\right]$$

$$= 2\mathrm{Re}\left[\frac{-j(\alpha^2 - \beta^2 + 2\alpha\beta j)}{\beta\beta_0{}^2 T(1 - e^{\alpha T}\cos\beta T - je^{\alpha T}\sin\beta T}\right]$$

$$= 2\mathrm{Re}\left[\frac{(\alpha + j\beta)^2\,[(-j)\,(1 - e^{\alpha T}\cos\beta T) - e^{\alpha T}\sin\beta T]}{\beta\beta_0{}^2 T\,[\,(1 - e^{\alpha T}\cos\beta T)^2 + e^{2\alpha T}\sin^2\beta T]}\right]$$

$$= 2\left[\frac{-(\alpha^2 - \beta^2)\,e^{\alpha T}\sin\beta T - 2\alpha\beta\,(e^{\alpha T}\cos\beta T - 1)}{\beta\beta_0{}^2 T\,[(1 - e^{\alpha T}\cos\beta T)^2 + e^{2\alpha T}\sin^2\beta T]}\right]$$

$$= \frac{2(\beta^2 - \alpha^2)\,e^{\alpha T}\sin\beta T + 4\alpha\beta\,(1 - e^{\alpha T}\cos\beta T)}{\beta\beta_0{}^2 T\,[1 + e^{2\alpha T} - 2e^{\alpha T}\cos\beta T]}$$

(2-22)

$$2\text{Re}\left[\frac{(\alpha+j\beta)^2\,(e^{-2(\alpha-j\beta)T}-1)}{8\beta^2 T\,(\alpha-j\beta)\,(e^{-(\alpha-j\beta)T}-1)^2}\right]$$

$$= 2\text{Re}\left[\frac{(\alpha+j\beta)^3\,[1+e^{-\alpha T}\cos\beta T+je^{\alpha T}\sin\beta T}{8\beta^2\beta_0{}^2 T\,[e^{-\alpha T}\cos\beta T+je^{-\alpha T}\sin\beta T-1]}\right]$$

$$= 2\text{Re}\left[\frac{(\alpha+j\beta)^3\,[e^{-2\alpha T}\cos^2\beta T-1+e^{-2\alpha T}\sin^2\beta T-2je^{-\alpha T}\sin\beta T]}{8\beta^2\beta_0{}^2 T\,[(e^{-\alpha T}\cos\beta T-1)^2+e^{-2\alpha T}\sin^2\beta T]}\right]$$

$$= 2\text{Re}\left[\frac{[\alpha^3-3\alpha\beta^2+j(3\alpha^2\beta-\beta^3)]\,[e^{-2\alpha T}-1-2je^{-\alpha T}\sin\beta T]}{8\beta^2\beta_0{}^2 T\,[e^{-2\alpha T}-2e^{-\alpha T}\cos\beta T+1]}\right]$$

$$= \frac{(\alpha^3-3\alpha\beta^2)\,(e^{-2\alpha T}-1)+2(3\alpha^2\beta-\beta^3)\,e^{-\alpha T}\sin\beta T}{4\beta^2\beta_0{}^2\,(e^{-2\alpha T}-2e^{-\alpha T}\cos\beta T+1)}$$

(2-23)

$$-\frac{\beta_0{}^2\,(e^{-2\alpha T}-1)}{4\alpha\beta^2 T\,(e^{-(\alpha-j\beta)T}-1)\,(e^{-(\alpha+j\beta)T}-1)}$$

$$= \frac{\beta_0{}^2\,(1-e^{-2\alpha T})}{4\alpha\beta^2 T\,[e^{-\alpha T}(\cos\beta T+j\sin\beta T)-1]\,[e^{-\alpha T}(\cos\beta T-j\sin\beta T)-1]}$$

$$= \frac{\beta_0{}^2\,(1-e^{-2\alpha T})}{4\alpha\beta^2 T\,[(e^{-\alpha T}\cos\beta T-1)^2+e^{-2\alpha T}\sin^2\beta T]}$$

$$= \frac{\beta_0^2 (1 - e^{-2\alpha T})}{4\alpha\beta^2 T\,[e^{-2\alpha T} - 2e^{-\alpha T} \cos \beta T + 1]}$$

(2-24)

Substituting the results of equations 2-20 through 2-24 into equation 2-19:

$$I = \frac{1}{3} - \frac{4\alpha}{\beta_0^2 T} + \frac{6\alpha^2 - 2\beta^2}{\beta_0^4 T}$$

$$+ \frac{2(\beta^2 - \alpha^2)\, e^{\alpha T} \sin \beta T + 4\alpha\beta (1 - e^{\alpha T} \cos \beta T)}{\beta\beta_0^2 T\,(1 + e^{2\alpha T} - 2e^{\alpha T} \cos \beta T)}$$

$$+ \frac{(\alpha^3 - 3\alpha\beta^2)(e^{-2\alpha T} - 1) + 2(3\alpha^2\beta - \beta^3)\, e^{-\alpha T} \sin \beta T}{4\beta^2 \beta_0^2 T\,(e^{-2\alpha T} - 2e^{-\alpha T} \cos \beta T + 1)}$$

$$+ \frac{\beta_0^2 (1 - e^{-2\alpha T})}{4\alpha\beta^2 T\,(e^{-2\alpha T} - 2e^{-\alpha T} \cos \beta T + 1)}$$

$$= \frac{1}{3} - \frac{4\alpha}{\beta_0^2 T} + \frac{6\alpha^2 - 2\beta^2}{\beta_0^4 T^2} + \frac{2(\beta^2 - \alpha^2) \sin \beta T + 4\alpha\beta (e^{-\alpha T} - \cos \beta T)}{2\beta\beta_0^2 T\,(\cosh \alpha T - \cos \beta T)}$$

$$+ \frac{(3\alpha\beta^2 - \alpha^3)\, 2 \sinh \alpha T + 2(3\alpha^2\beta - \beta^3) \sin \beta T}{8\beta^2 \beta_0^2 \cdot T(\cosh \alpha T - \cos \beta T)}$$

$$+ \frac{2\beta_0^{\,2} \sinh \alpha T}{8\alpha\beta^2 T\,(\cosh \alpha T - \cos \beta T)}$$

(2-25)

Simplifying,

$$I = \frac{1}{3} - \frac{4\alpha}{\beta_0^{\,2} T} + \frac{6\alpha^2 - 2\beta^2}{\beta_0^{\,4} T^2}$$

$$+ \frac{(\beta^3 + 5\alpha^2\beta) \sinh \alpha T + (3\alpha\beta^2 - \alpha^3) \sin \beta T + 8\alpha^2\beta\,(e^{-\alpha T} - \cos \beta T)}{4\alpha\beta\beta_0^{\,2} T\,(\cosh \alpha T - \cos \beta T)}$$

(2-26)

Equation 2-26 expresses the error squared integral I as a function of three variables; α, β, and T. It can now conveniently be transformed into a function of the scaled time variable αT and damping ratio ζ. The two forms of system characteristic equation can be compared below.

$$S^2 + 2\zeta W_0 S + W_0^{\,2} = 0$$

$$S^2 + 2\alpha S + \alpha^2 + \beta^2 = 0$$

It can be seen that

$$W_0^{\,2} = \alpha^2 + \beta^2 \quad \text{and} \quad \alpha^2 = \zeta^2 W_0^{\,2}.$$

Then

$$\beta^2 = W_0^{\ 2}\ (1 - \zeta^2) = \alpha^2\ \frac{1 - \zeta^2}{\zeta^2} \quad .$$

Let

$$\frac{1 - \zeta^2}{\zeta^2} = A^2 \ . \tag{2-27}$$

If the numerator and denominator of each term of equation 2-26 are multiplied by the appropriate power of T and the substitutions

$$\alpha T = Z \quad \text{and} \quad \beta T = AZ \tag{2-28}$$

are made, I takes the form:

$$I = \frac{1}{3} - \frac{4Z\zeta^2}{Z^2} + \frac{(6Z^2 - 2A^2Z^2)\,\zeta^4}{Z^4}$$

$$+ \frac{(A^3Z^3 + 5AZ^3)\sinh Z + (3A^2Z^3 - Z^3)\sin AZ + 8AZ^3(e^{-Z} - \cos AZ)}{4AZ^4\,(1 + A^2)\,(\cosh Z - \cos AZ)}$$

$$= \frac{1}{3} - \frac{4\zeta^2}{Z} + \frac{2\zeta^4(3 - A^2)}{Z^2}$$

$$+ \frac{(A^3 + 5A) \sinh Z + (3A^2 - 1) \sin AZ + 8A\,(e^{-Z} - \cos AZ)}{\frac{4AZ}{\zeta^2}\,(\cosh Z - \cos AZ)}$$

(2-29)

From equation 2-27

$$1 + A^2 = \frac{1}{\zeta^2}$$

therefore

$$3 - A^2 = 4 - (1 + A^2) = \frac{4\zeta^2 - 1}{\zeta^2} \quad ,$$

$$A^3 + 5A = A(1 + A^2 + 4) = A\,\frac{4\zeta^2 + 1}{\zeta^2} \quad ,$$

and

$$3A^2 - 1 = 3(A^2 + 1) - 4 = \frac{3 - 4\zeta^2}{\zeta^2} \quad .$$

Finally, using these relationships in equation 2-29,

$$I = \frac{1}{3} - \frac{4\zeta^2}{Z} + \frac{2\zeta^2(4\zeta^2 - 1)}{Z^2}$$

$$+ \frac{(4\zeta^2 + 1) \sinh Z + (3 - 4\zeta^2)\,(\sin AZ)\,/A + 8\zeta^2\,(e^{-Z} - \cos AZ)}{4Z\,(\cosh Z - \cos AZ)}$$

(2-30)

DERIVATION OF THE ERROR SQUARED INTEGRAL FOR THE THIRD ORDER SYSTEM

The transfer function for a third order underdamped system can be written in the form:

$$\frac{\theta_1(S)}{R(S)} = \frac{\gamma\beta_0^{\,2}}{(S+\gamma)\,[(S+\alpha)^2+\beta^2]} \tag{3-1}$$

where α, β, and γ are constants, $\beta_0^{\,2} = \alpha^2 + \beta^2$, and R(S) is defined by equation 2-2.

Then

$$\theta_1(S) = \frac{\gamma\beta_0^{\,2}}{(S+\gamma)\,[(S+\alpha)^2+\beta^2]} \cdot \frac{1}{S}\,[1+e^{-ST}+e^{-2ST}+\,.\,.\,.\,.\,.\,.\,]$$

and

$$\theta_1(t) = 1 - \frac{\beta_0^{\,2}}{\gamma_0^{\,2}}\,e^{-\gamma t} + \frac{\gamma\beta_0}{\beta\gamma_0}\,e^{-\alpha t}\sin[\beta t - \phi_1]$$

$$+\,1 - \frac{\beta_0^{\,2}}{\gamma_0^{\,2}}\,e^{-\gamma(t-T)} + \frac{\gamma\beta_0}{\beta\gamma_0}\,e^{-\alpha(t-T)}\sin[\beta(t-T) - \phi_1]$$

$+ -$

$- -$

$- -$

$$+ 1 - \frac{\beta_0^2}{\gamma_0^2} e^{-\gamma(t - KT)} + \frac{\gamma\beta_0}{\beta\gamma_0} e^{-\alpha(t - KT)} \sin[\beta(t - KT) - \phi_1] . \tag{3-2}$$

$$\theta_1(t) = K + 1 - \frac{\beta_0^2}{\gamma_0^2} \sum_{k=0}^{K} e^{-\gamma(t - kT)} + \frac{\gamma\beta_0}{\beta\gamma_0} \sum_{k=0}^{K} e^{-\alpha(t - kT)} \sin[\beta(t - kT) - \phi_1] ,$$

$$\text{where } \gamma_0^2 = (\alpha - \gamma)^2 + \beta^2 \text{ and } \phi_1 = \tan^{-1}\frac{\beta}{-\alpha} + \tan^{-1}\frac{\beta}{\gamma - \alpha} . \tag{3-3}$$

$$\sum_{k=0}^{K} e^{-\gamma(t - kT)} = e^{-\gamma t} \sum_{k=0}^{K} e^{\gamma kT} = \frac{e^{-\gamma t}\,(1 - e^{(K+1)\gamma T})}{1 - e^{\gamma T}} \tag{3-4}$$

From equations 2-5 through 2-9 it can be seen that

$$\sum_{k=0}^{K} e^{-\alpha(t - kT)} \sin[\beta(t - kT) - \phi_1]$$

$$= \frac{1}{2j} \left[e^{-j\phi_1} \cdot e^{-KT(\alpha - j\beta)} \cdot e^{-x(\alpha - j\beta)} \frac{1 - e^{(K+1)(\alpha - j\beta)T}}{1 - e^{(\alpha - j\beta)T}} \right.$$

$$\left. - e^{j\phi_1} \cdot e^{-KT(\alpha + j\beta)} \cdot e^{-x(\alpha + j\beta)} \frac{1 - e^{(K+1)(\alpha + j\beta)T}}{1 - e^{(\alpha + j\beta)T}} \right],$$

(3-5)

where x is defined by equation 2-8.

Let $\tan^{-1} \frac{\beta}{-\alpha} = \alpha_1$ and $\tan^{-1} \frac{\beta}{\gamma - \alpha} = \beta_1$. Then from equation 3-3

$$e^{j\phi_1} = \cos(\alpha_1 + \beta_1) + j \sin(\alpha_1 + \beta_1)$$

$$= \cos\alpha_1 \cos\beta_1 - \sin\alpha_1 \sin\beta_1 + j(\sin\alpha_1 \cos\beta_1 + \cos\alpha_1 \sin\beta_1)$$

$$= -\frac{\alpha}{\beta_0} \cdot \frac{(\gamma - \alpha)}{\gamma_0} - \frac{\beta}{\beta_0} \cdot \frac{\beta}{\gamma_0} + j\left[\frac{\beta}{\beta_0} \cdot \frac{(\gamma - \alpha)}{\gamma_0} + \left(\frac{-\alpha}{\beta_0} \cdot \frac{\beta}{\gamma_0}\right)\right]$$

$$= \frac{1}{\beta_0\gamma_0} \left[-\alpha(\gamma - \alpha) - \beta^2 + j(\beta[\gamma - \alpha] - \alpha\beta)\right]$$

(3-6)

and

$$e^{-j\phi_1} = \frac{1}{\beta_0\gamma_0} \left[-\alpha(\gamma - \alpha) - \beta^2 - j(\beta[\gamma - \alpha] - \alpha\beta)\right].$$

(3-7)

In terms of x, the right hand side of equation 3-4 becomes

$$\frac{e^{-\gamma(KT+x)}\,(1-e^{(K+1)\gamma T})}{1-e^{\gamma T}} = \frac{e^{-\gamma(KT+x)} - e^{\gamma(T-x)}}{1-e^{\gamma T}} \quad . \tag{3-8}$$

By using equations 3-4 through 3-8 and taking K large in equations 3-5 and 3-8, equation 3-3 can now be transformed into a steady state response function for the third order system.

$$\theta_1(x) = K+1 - \frac{\beta_0^{\,2}}{\gamma_0^{\,2}}\left[\frac{-e^{\gamma(T-x)}}{1-e^{\gamma T}}\right]$$

$$+ \frac{\gamma\beta_0}{2j\beta\gamma_0}\left[\frac{1}{\beta_0\gamma_0}\right]\left[\left(-\alpha(\gamma-\alpha)-\beta^2 - j(\beta[\gamma-\alpha]-\alpha\beta)\right)e^{-x(\alpha-j\beta)}\right.$$

$$\cdot\frac{-e^{(\alpha-j\beta)T}}{1-e^{(\alpha-j\beta)T}} - \left(-\alpha(\gamma-\alpha)-\beta^2 + j(\beta[\gamma-\alpha]-\alpha\beta)\right)e^{-x(\alpha+j\beta)}$$

$$\left.\cdot\frac{-e^{(\alpha+j\beta)T}}{1-e^{(\alpha-j\beta)T}}\right] \tag{3-9}$$

Let

$$\alpha(\gamma-\alpha)+\beta^2 = M \quad , \quad 2\alpha\beta - \beta\gamma = N,$$

$$\frac{M - jN}{e^{-(\alpha - j\beta)T} - 1} = D \, , \quad \text{and} \quad \frac{M + jN}{e^{-(\alpha + j\beta)T} - 1} = \overline{D} \tag{3-10}$$

Using equations 3-10 and 2-14, the error squared integral for the third order system can be expressed as follows:

$$I_1 = \frac{1}{T} \int_0^T \left\{ 1 - \frac{x}{T} + \frac{\beta_0^2}{\gamma_0^2} \frac{e^{-\gamma x}}{e^{-\gamma T} - 1} + \frac{\gamma}{2j\beta\gamma_0^2} \left[De^{-x(\alpha - j\beta)} - \overline{D}e^{-x(\alpha + j\beta)} \right] \right\}^2 dx \tag{3-11}$$

Squaring the integrand:

$$I_1 = \frac{1}{T} \int_0^T \left\{ 1 + \frac{x^2}{T^2} + \frac{\beta_0^4}{\gamma_0^4} \frac{e^{-2\gamma x}}{(e^{-\gamma T} - 1)} - \frac{\gamma^2}{4\beta^2\gamma_0^4} \left[D^2 e^{-2x(\alpha - j\beta)} - 2D\overline{D}e^{-2\alpha x} + \overline{D}^2 e^{-2x(\alpha + j\beta)} \right] \right.$$

$$- \frac{2x}{T} + \frac{2\beta_0^2 e^{-\gamma x}}{\gamma_0^2 (e^{-\gamma T} - 1)} + \frac{2\gamma}{2j\beta\gamma_0^2} \left[De^{-x(\alpha - j\beta)} - \overline{D}e^{-x(\alpha + j\beta)} \right]$$

$$- \frac{2\beta_0^{\;2} \; x \; e^{-\gamma x}}{T\gamma_0^{\;2} \; (e^{-\gamma T} - 1)} - \frac{\gamma x}{jT\beta\gamma_0^{\;2}} \left[De^{-x(\alpha - j\beta)} - \bar{D}e^{-x(\alpha + j\beta)} \right]$$

$$+ \frac{\beta_0^{\;2} \; \gamma e^{-\gamma x}}{j\beta\gamma_0^{\;4} \; (e^{-\gamma T} - 1)} \left[De^{-x(\alpha - j\beta)} - \bar{D}e^{-x(\alpha + j\beta)} \right] \Bigg\} dx \tag{3-12}$$

Integrating equation 3-12 with the aid of equations 2-16 and 2-17:

$$I_1 = 1 + \frac{1}{3} - \frac{\beta_0^{\;4} \; (e^{-2\gamma T} - 1)}{2T\gamma\gamma_0^{\;4} \; (e^{-\gamma T} - 1)^2} + \frac{\gamma^2}{4T\beta^2 \; \gamma_0^{\;4}}$$

$$\left[\frac{D^2 \; (e^{-2(\alpha - j\beta)T} - 1)}{2(\alpha - j\beta)} - \frac{D\bar{D}}{\alpha} \; (e^{-2\alpha T} - 1) + \frac{\bar{D}^2 \; (e^{-2(\alpha + j\beta)T} - 1)}{2(\alpha + j\beta)} \right]$$

$$- 1 + \frac{-2\beta_0^{\;2}}{T\gamma\gamma_0^{\;2}} + \frac{\gamma}{jT\beta\gamma_0^{\;2}} \left[\frac{D(e^{-(\alpha - j\beta)T} - 1)}{-(\alpha - j\beta)} + \frac{\bar{D} \; (e^{-(\alpha + j\beta)T} - 1)}{(\alpha + j\beta)} \right]$$

$$+ \frac{2\beta_0^{\;2}}{T^2 \; \gamma_0^{\;2} \; (e^{-\gamma T} - 1)} \left[\frac{T}{\gamma} \; e^{-\gamma T} + \frac{1}{\gamma^2} \; (e^{-\gamma T} - 1) \right]$$

$$+ \frac{\gamma}{jT^2\beta\gamma_0^{\;2}} \left[D \left(\frac{T(e^{-(\alpha - j\beta)T})}{\alpha - j\beta} + \frac{(e^{-(\alpha - j\beta)T} - 1)}{(\alpha - j\beta)^2} \right) \right.$$

$$- \bar{D} \left(\frac{Te^{-(\alpha + j\beta)T}}{\alpha + j\beta} + \frac{e^{-(\alpha + j\beta)T} - 1}{(\alpha + j\beta)^2} \right) \Bigg] + \frac{\beta_0{}^2 \gamma}{jT\beta\gamma_0{}^4 (e^{-\gamma T} - 1)}$$

$$\left[\frac{D(e^{-(\alpha + \gamma - j\beta)T} - 1)}{-(\alpha + \gamma - j\beta)} + \frac{\bar{D}(e^{-(\alpha + \gamma + j\beta)T} - 1)}{\alpha + \gamma + j\beta} \right] \tag{3-13}$$

Equation 3-13 can be simplified by combining terms which appear in conjugate pairs.

$$- \frac{\beta_0{}^4 (e^{-\gamma T} + 1)(e^{-\gamma T} - 1)}{2T\gamma\gamma_0{}^4 (e^{-\gamma T} - 1)^2} = \frac{\beta_0{}^4}{2T\gamma\gamma_0{}^4} \coth \frac{\gamma T}{2} \tag{3-14}$$

$$\frac{\gamma^2}{4T\beta^2\gamma_0{}^4} \cdot 2\mathrm{Re} \left[\frac{(M^2 - N^2 - 2jMN)(e^{-2(\alpha - j\beta)T} - 1)}{2(\alpha - j\beta)(e^{-(\alpha - j\beta)T} - 1)^2} \right]$$

$$= \frac{\gamma^2}{4T\beta^2\gamma_0{}^4} \mathrm{Re} \left[\frac{(M^2 - N^2 - 2jMN)(e^{-\alpha T} \cos \beta T + 1 + je^{-\alpha T} \sin \beta T)}{(\alpha - j\beta)(e^{-\alpha T} \cos \beta T - 1 + je^{-\alpha T} \sin \beta T)} \right]$$

$$= \frac{\gamma^2 \mathrm{Re} \left\{ [\alpha(M^2 - N^2) + 2\beta MN + j(\beta(M^2 - N^2) - 2\alpha MN)] \right.}{4T\beta^2\gamma_0{}^4 \beta_0{}^2 [(e^{-\alpha T} \cos \beta T - 1)^2 + e^{-2\alpha T} \sin^2 \beta T]} *$$

Continuation of numerator

$$* \left. [e^{-2\alpha T} \cos^2 \beta T - 1 - 2je^{-\alpha T} \sin \beta T + e^{-2\alpha T} \sin^2 \beta T] \right\}$$

$$= \frac{\gamma^2 \operatorname{Re}\{[\alpha(M^2 - N^2) + 2\beta MN + j(\beta(M^2 - N^2) - 2\alpha MN)]}{4T\beta^2 \beta_0^2 \gamma_0^4 [e^{-2\alpha T} - 2e^{-\alpha T}\cos\beta T + 1]} *$$

$$* \underline{[(e^{-2\alpha T} - 1) - 2je^{-\alpha T}\sin\beta T]\}}$$

$$= \frac{\gamma^2\{[\alpha(M^2 - N^2) + 2\beta MN][-\sinh\alpha T] + \sin\beta T\,[\beta(M^2 - N^2) - 2\alpha MN]\}}{4T\beta^2 \beta_0^2 \gamma_0^4 (\cosh\alpha T - \cos\beta T)}$$

(3-15)

$$- \frac{\gamma^2 D\bar{D}(e^{-2\alpha T} - 1)}{4T\alpha\beta^2 \gamma_0^4} = - \frac{\gamma^2(M^2 + N^2)(e^{-2\alpha T} - 1)}{4T\alpha\beta^2 \gamma_0^4 (e^{-\alpha T}\cos\beta T - 1)^2 + e^{-2\alpha T}\sin^2\beta T}$$

$$= \frac{\gamma^2 (M^2 + N^2)\sinh\alpha T}{4T\alpha\beta^2 \gamma_0^4 (\cosh\alpha T - \cos\beta T)}$$

(3-16)

$$\frac{\gamma}{T\beta\gamma_0^2} \cdot 2\operatorname{Im}\left\{\frac{D(e^{-(\alpha - j\beta)T} - 1)}{-(\alpha - j\beta)}\right\}$$

$$= \frac{\gamma}{T\beta\gamma_0^2} \cdot 2\operatorname{Im}\left\{\frac{(M - jN)(e^{-(\alpha - j\beta)T} - 1)}{-(\alpha - j\beta)(e^{-(\alpha - j\beta)T} - 1)}\right\}$$

$$= \frac{2\gamma}{T\beta\gamma_0^{\ 2}} \cdot \mathrm{Im}\left\{-(\alpha M + \beta N) + j(\alpha N - \beta M)\right\} = \frac{2\gamma(\alpha N - \beta M)}{T\beta\gamma_0^{\ 2}\ \beta_0^{\ 2}}$$

(3-17)

$$\frac{2\beta_0^{\ 2}}{T^2\ \gamma_0^{\ 2}\ (e^{-\gamma T}-1)} \left[\frac{Te^{-\gamma T}}{\gamma} + \frac{e^{-\gamma T}-1}{\gamma^2}\right] = \frac{2\beta_0^{\ 2}}{T^2\gamma_0^{\ 2}} \left[\frac{T}{\gamma(1-e^{\gamma T})} + \frac{1}{\gamma^2}\right]$$

(3-18)

$$\frac{\gamma}{T^2\beta\gamma_0^{\ 2}} \cdot 2\,\mathrm{Im}\left\{\frac{(M-jN)\,Te^{-(\alpha-j\beta)T}}{(e^{-(\alpha-j\beta)T}-1)(\alpha-j\beta)} + \frac{(M-jN)\,(e^{-(\alpha-j\beta)T}-1)}{(\alpha-j\beta)^2\,(e^{-(\alpha-j\beta)T}-1)}\right\}$$

$$= \frac{2\gamma}{T^2\beta\gamma_0^{\ 2}}\ \mathrm{Im}\left\{\frac{T(M-jN)\,(\alpha+j\beta)}{\beta_0^{\ 2}[1-(e^{\alpha T}\cos\beta T - je^{\alpha T}\sin\beta T)]}\right.$$

$$\left. + \frac{(M-jN)\,(\alpha^2 - \beta^2 + 2j\alpha\beta)}{\beta_0^{\ 4}}\right\}$$

$$= \frac{2\gamma}{T^2\beta\gamma_0^{\ 2}} \left[\frac{2\alpha\beta M - N(\alpha^2-\beta^2)}{\beta_0^{\ 4}} + \mathrm{Im}\left\{\frac{T[\alpha M + \beta N + j(\beta M - \alpha N)]}{\beta_0^{\ 2}[1 - e^{\alpha T}\cos\beta T + je^{\alpha T}\sin\beta T}\right\}\right]$$

$$= \frac{2\gamma}{T^2\beta\gamma_0^{\ 2}\ \beta_0^{\ 2}} \left[T\left[\frac{(\beta M - \alpha N)\,(1-e^{\alpha T}\cos\beta T) - (\alpha M - \beta N)(e^{\alpha T}\sin\beta T)}{(1-e^{\alpha T}\cos\beta T)^2 + e^{2\alpha T}\sin^2\beta T}\right]\right.$$

$$+ \frac{2\alpha\beta M - N(\alpha^2 - \beta^2)}{\beta_0^2}\Bigg]$$

$$= \frac{2\gamma}{T^2\, \beta\gamma_0^2\, \beta_0^2} \left[\frac{T[(\beta M - \alpha N)(1 - e^{\alpha T}\cos\beta T - (\alpha M + \beta N)e^{\alpha T}\sin\beta T]}{2e^{\alpha T}(\cosh\alpha T - \cos\beta T)} \right.$$

$$\left. + \frac{2\alpha\beta M - N(\alpha^2 - \beta^2)}{\beta_0^2} \right]$$

(3-19)

$$\frac{\beta_0^2\gamma}{T\beta\gamma_0^4(e^{-\gamma T} - 1)} \cdot 2\,\mathrm{Im}\left[\frac{D(e^{-(\alpha+\gamma-j\beta)T} - 1)}{-(\alpha+\gamma-j\beta)}\right]$$

$$= \frac{2\beta_0^2\gamma}{T\beta\gamma_0^4(e^{-\gamma T} - 1)}\,\mathrm{Im}\left[\frac{(M - jN)(e^{-(\alpha+\gamma)T}\cos\beta T - 1 + je^{-(\alpha+\gamma)T}\sin\beta T)}{-(\alpha+\gamma-j\beta)(e^{-\alpha T}\cos\beta T - 1 + je^{-\alpha T}\sin\beta T)}\right]$$

$$= \frac{-2\beta_0^2\gamma}{T\beta\gamma_0^4(e^{-\gamma T} - 1)}\,\mathrm{Im}\left[\frac{[M(\alpha+\gamma) + N\beta + j[\beta M - N(\alpha+\gamma)]]\,[e^{-(2\alpha+\gamma)T}}{[(\alpha+\gamma)^2 + \beta^2]\,[(e^{-\alpha T}\cos\beta T - 1)^2 + e^{-2\alpha T}\sin^2\beta T} \right. {}^{*}$$

$$\left. {}^{*}\ \frac{+1 - (1 + e^{-\gamma T})\,e^{-\alpha T}\cos\beta T + j(1 - e^{-\gamma T})\,e^{-\alpha T}\sin\beta T]}{} \right]$$

$$= \frac{-\gamma\beta_0^{\,2}}{T\beta\gamma_0^{\,4}(e^{-\gamma T}-1)} \left[\frac{[M(\alpha+\gamma)+N\beta]\,[(1-e^{-\gamma T})\, e^{-\alpha T} \sin \beta T] + }{[(\alpha+\gamma)^2+\beta^2]\;\; 2e^{-\alpha T}[\cosh \alpha T - \cos \beta T]} \right. *$$

$$\left. * \frac{[\beta M - N(\alpha+\gamma)]\,[e^{-(2\alpha+\gamma)T} + 1 - e^{-\alpha T}(1+e^{-\gamma T}) \cos \beta T]}{} \right]$$

$$= \frac{-\gamma\beta_0^{\,2}}{T\beta\gamma_0^{\,4}} \left[\frac{[M(\alpha+\gamma)+N\beta]\;[-\sin\beta T] + [\beta M - N(\alpha+\gamma)]}{[(\alpha+\gamma)^2+\beta^2]\;[\cosh\alpha T - \cos\beta T]} \right. *$$

$$\left. * \frac{[\cos\beta T \coth \frac{\gamma T}{2} + (e^{-(\alpha+\gamma)} + e^{\alpha T})\,/(e^{-\gamma T}-1)}{} \right] \tag{3-20}$$

Substituting the results of equations 3-14 through 3-20 into equation 3-13:

$$I_1 = \frac{1}{3} + \frac{\beta_0^{\,4}}{2T\gamma\gamma_0^{\,4}} \coth\frac{\gamma T}{2} + \frac{\gamma^2}{4T\beta^2\beta_0^{\,2}\gamma_0^{\,4}}$$

$$\left[\frac{[\beta(M^2-N^2) - 2\alpha MN]\;\sin\beta T - [\alpha(M^2-N^2)+2\beta MN]\;\sinh\alpha T}{\cosh\alpha T - \cos\beta T} \right]$$

$$+ \frac{\gamma^2(M^2+N^2)\sinh\alpha T}{4T\alpha\beta^2\gamma_0^{\,4}(\cosh\alpha T - \cos\beta T)} - \frac{2\beta_0^{\,2}}{T\gamma\gamma_0^{\,2}} + \frac{2\gamma(\alpha N - \beta M)}{T\beta\gamma_0^{\,2}\beta_0^{\,2}}$$

$$+ \frac{2\beta_0^2}{T^2\gamma_0^2}\left[\frac{T}{\gamma(1-e^{\gamma T})}+\frac{1}{\gamma^2}\right] + \frac{2\gamma}{T^2\beta\beta_0^2\gamma_0^2}\left[\frac{2\alpha\beta M - N(\alpha^2-\beta^2)}{\beta_0^2}\right.$$

$$\left. + \frac{T[(\beta M - \alpha N)(e^{-\alpha T} - \cos\beta T) - (\alpha M + \beta N)\sin\beta T]}{2(\cosh\alpha T - \cos\beta T)}\right]$$

$$- \frac{\gamma\beta_0^2}{T\beta\gamma_0^4}\left[\frac{-[M(\alpha+\gamma)+\beta N]\sin\beta T + [\beta M - N(\alpha+\gamma)]}{[(\alpha+\gamma)^2+\beta^2][\cosh\alpha T - \cos\beta T]}\right. *$$

$$\left. * \underline{[\cos\beta T\coth\frac{\gamma T}{2} + (e^{-\alpha T} + e^{(\alpha+\gamma)T})/(1-e^{\gamma T})]}\right] \qquad (3\text{-}21)$$

Let $\alpha T = Z$ and $\beta T = AZ$ as in the case of the second order system where A is defined by equation 2-27. In addition let

$$\gamma T = \lambda Z \qquad (3\text{-}22)$$

Then from equations 3-1, 3-3, and 3-10:

$$MT^2 = \alpha T(\gamma T - \alpha T) + \beta^2 T^2 = Z^2(\lambda - 1 + A^2),$$

$$NT^2 = 2\alpha\beta T^2 - \beta\gamma T^2 = Z^2(2 - \lambda)$$

$$\gamma_0^2 T^2 = (\alpha T - \gamma T)^2 + \beta^2 T^2 = Z^2[(1-\lambda)^2 + A^2],$$

and

$$\beta_0{}^2T^2 = a^2T^2 + \beta^2T^2 = Z^2(1 + A^2) = Z^2/\zeta^2$$

(3-23)

Using equations 2-27, 2-28, 3-22, and 3-23, equation 3-21 can be expressed as a function of ζ, λ, and Z.

$$\frac{\beta_0{}^4}{2\gamma\gamma_0{}^4T} \coth\frac{\gamma T}{2} = \frac{Z^4/\zeta^4 \coth\frac{\lambda Z}{2}}{2\lambda Z\cdot Z^4[(1-\lambda)^2 + A^2]}$$

$$= \frac{\coth\frac{\lambda Z}{2}}{2\lambda Z\zeta^4[(1-\lambda)^2 + A^2]^2}$$

(3-24)

$$\frac{\gamma^2[\beta(M^2 - N^2) - 2aMN]}{4T\beta^2\beta_0{}^2\gamma_0{}^4}$$

$$= \frac{\lambda^2Z^2[AZ(Z^4[\lambda - 1 + A^2]^2 - A^2Z^4[2-\lambda]^2) - 2Z^3[\lambda - 1 + A^2]}{4A^2Z^2\cdot Z^2\cdot Z^4[(1-\lambda)^2 + A^2]^2\,(1/\zeta^2)} *$$

$$* \quad \underline{\cdot AZ^2[2-\lambda]]}$$

$$= \frac{\zeta^2\lambda^2[(\lambda-1+A^2)^2 - A^2(2-\lambda)^2 - 2(\lambda-1+A^2)(2-\lambda)]}{4AZ[(1-\lambda)^2 + A^2]^2}$$

(3-25)

$$\frac{-\gamma^2[\alpha(M^2-N^2) + 2\beta MN}{4T\beta^2\beta_0{}^2\gamma_0{}^4}$$

$$= \frac{\lambda^2 Z^2 [Z(Z^4[\lambda-1+A^2]^2 - A^2Z^4[2-\lambda]^2) + 2AZ^3(\lambda-1+A^2)}{-4A^2Z^2(Z^2/\zeta^2)Z^4[(1-\lambda)^2 + A^2]^2} *$$

$$* \underline{\cdot AZ^2[2-\lambda]]}$$

$$= - \frac{\zeta^2\lambda^2[(\lambda-1+A^2)^2 - A^2(2-\lambda)^2 + 2A^2(2-\lambda)(\lambda-1+A^2)]}{4A^2Z[(1-\lambda)^2 + A^2]^2}$$

(3-26)

$$\frac{\gamma^2(M^2+N^2)}{4\alpha T\beta^2\gamma_0{}^4} = \frac{\lambda^2 Z^2[Z^4(\lambda-1+A^2)^2 + A^2Z^4(2-\lambda)^2]}{4Z\cdot A^2Z^2\cdot Z^4[(1-\lambda)^2 + A^2]^2}$$

$$= \frac{\lambda^2[(\lambda-1+A^2)^2 + A^2(2-\lambda)^2]}{4A^2Z[(1-\lambda)^2 + A^2]^2}$$

(3-27)

$$-\frac{2\beta_0^{\,2}}{T\gamma\gamma_0^{\,2}} = -\frac{2Z^2/\zeta^2}{\lambda Z^3\,[(1-\lambda)^2 + A^2]} = \frac{-2}{\lambda\zeta^2 Z\,[(1-\lambda)^2 + A^2]}$$

(3-28)

$$\frac{2\gamma\,(\alpha N - \beta M)}{\beta\,T\gamma_0^{\,2}\,\beta_0^{\,2}} = \frac{2\,\lambda Z\,[AZ^3\,(2-\lambda) - AZ^3\,(\lambda - 1 + A^2)]}{AZ^3\,(Z^2/\zeta^2)\,[(1-\lambda)^2 + A^2]}$$

$$= \frac{2\lambda\zeta^2\,[2 - 2\lambda + 1 - A^2]}{Z[(1-\lambda)^2 + A^2]} = \frac{2\,\lambda\zeta^2\,[3 - 2\lambda - A^2]}{Z[(1-\lambda)^2 + A^2]}$$

(3-29)

$$\frac{2\beta_0^{\,2}}{T^2\gamma_0^{\,2}}\left[\frac{T}{\gamma(1-e^{\gamma T})} + \frac{1}{\gamma^2}\right]$$

$$= \frac{2Z^2/\zeta^2}{Z^2\,[(1-\lambda)^2 + A^2]}\left[\frac{1}{\lambda Z(1-e^{\lambda Z})} + \frac{1}{\lambda^2 Z^2}\right]$$

$$= \frac{2}{\zeta^2\lambda Z\,[(1-\lambda)^2 + A^2]}\left[\frac{1}{1-e^{\lambda Z}} + \frac{1}{\lambda Z}\right]$$

(3-30)

$$\frac{2\gamma\,[2\alpha\beta M - N\,(\alpha^2 - \beta^2)]}{T^2\beta\beta_0^{\,2}\,\gamma_0^{\,2}\,\beta_0^{\,2}}$$

$$= \frac{2\lambda Z\,[2Z\cdot AZ\cdot Z^2\,[\lambda - 1 + A^2] - AZ^2\,(2-\lambda)\,(Z^2 - A^2Z^2)]}{AZ\,(Z^4/\zeta^4)\,Z^2\,[(1-\lambda)^2 + A^2]}$$

$$= \frac{2\zeta^4\lambda\,[2(\lambda - 1 + A^2) - (2-\lambda)\,(1 - A^2)]}{Z^2\,[(1-\lambda)^2 + A^2]}$$

(3-31)

$$\frac{2\gamma T\,(\beta M - \alpha N)}{2T^2\,\beta\beta_0{}^2\,\gamma_0{}^2} = \frac{\lambda Z\,[AZ^3\,(\lambda - 1 + A^2) - AZ^3\,(2-\lambda)]}{AZ\,(Z^2/\zeta^2)\,Z^2\,[(1-\lambda)^2 + A^2]}$$

$$= \frac{\lambda\zeta^2 A\,[2\lambda - 3 + A^2]}{AZ\,[(1-\lambda)^2 + A^2]}$$

(3-32)

$$\frac{-2\gamma T\,(\alpha M + \beta N)}{2T^2\,\beta\beta_0{}^2\,\gamma_0{}^2} = \frac{-\lambda Z\,[Z^3\,(\lambda - 1 + A^2) + A^2Z^3\,(2-\lambda)]}{AZ(Z^2/\zeta^2)\,Z^2\,[(1-\lambda)^2 + A^2]}$$

$$= -\frac{\zeta^2\lambda[\lambda - 1 + 3A^2 - 2A^2\lambda]}{AZ\,[(1-\lambda)^2 + A^2]}$$

(3-33)

$$\frac{\gamma\beta_0{}^2}{T\beta\gamma_0{}^4} \cdot \frac{M(\alpha + \gamma) + N\beta}{(\alpha + \gamma)^2 + \beta^2}$$

$$= \frac{\lambda Z (Z^2/\zeta^2) [Z^2(\lambda - 1 + A^2)(Z + \lambda Z) + A^2 Z^3 (2 - \lambda)]}{AZ^5 [(1-\lambda)^2 + A^2]^2 [(Z + \lambda Z)^2 + A^2 Z^2]}$$

$$= \frac{\lambda Z [(\lambda - 1 + A^2)(1 + \lambda) + A^2 (2 - \lambda)]}{\zeta^2 A [(1-\lambda)^2 + A^2]^2 Z^2 [(1+\lambda)^2 + A^2]}$$

(3-34)

$$\frac{-\gamma\beta_0^2 [\beta M - N(\alpha + \gamma)]}{T\beta\gamma_0^4 [(\alpha + \gamma)^2 + \beta^2]}$$

$$= \frac{-\lambda Z (Z^2/\zeta^2) [AZ^3 (\lambda - 1 + A^2) - AZ^2 (2 - \lambda)(Z - \lambda Z)]}{AZ \cdot Z^4 [(1-\lambda)^2 + A^2]^2 [(Z + \lambda Z)^2 + A^2 Z^2]}$$

$$= - \frac{A\lambda Z [\lambda^2 + A^2 - 3]}{A \zeta^2 [(1-\lambda)^2 + A^2]^2 Z^2 [(1+\lambda)^2 + A^2]}$$

(3-35)

Substituting the results of equations 3-24 through 3-35 into equation 3-21:

$$I_1 = \frac{1}{3} + \frac{\coth(\lambda Z/2)}{2\lambda Z \zeta^4 [(1-\lambda)^2 + A^2]^2}$$

$$+ \frac{A\zeta^2\lambda^2 [(\lambda - 1 + A^2)^2 - A^2 (2-\lambda)^2 - 2(\lambda - 1 + A^2)(2-\lambda)] \sin \lambda Z}{4A^2 Z [(1-\lambda)^2 + A^2]^2 [\cosh Z - \cos AZ]} \; *$$

$$* \; \frac{-\zeta^2\lambda^2 [(\lambda - 1 + A^2)^2 - A^2 (2-\lambda)^2 + 2A^2 (2-\lambda)(\lambda - 1 + A^2)] \sinh Z}{}$$

$$+ \frac{\lambda^2 [(\lambda - 1 + A^2)^2 + A^2 (2-\lambda)^2] \sinh Z}{4A^2 Z [(1-\lambda)^2 + A^2]^2 [\cosh Z - \cos AZ]}$$

$$- \frac{2}{\lambda \zeta^2 Z [(1-\lambda)^2 + A^2]} + \frac{2\lambda \zeta^2 (3 - 2\lambda - A^2)}{Z [(1-\lambda)^2 + A^2]}$$

$$+ \frac{2}{\lambda Z \zeta^2 [(1-\lambda)^2 + A^2]} \left[\frac{1}{1 - e^{\lambda Z}} + \frac{1}{\lambda Z} \right]$$

$$+ \frac{2\lambda \zeta^4 [2(\lambda - 1 + A^2) - (2-\lambda)(1 - A^2)]}{Z^2 [(\lambda - 1)^2 + A^2]}$$

$$+ \frac{\zeta^2 A\lambda (A^2 - 3 + 2\lambda)(e^{-Z} - \cos AZ) - \zeta^2 \lambda (\lambda - 1 + 3A^2 - A^2\lambda) \sin AZ}{AZ [(1-\lambda)^2 + A^2] [\cosh Z - \cos AZ]}$$

$$+ \frac{\lambda (3A^2 + \lambda - 1) \sin AZ - A\lambda [A^2 + \lambda^2 - 3] [\cos AZ \coth (\lambda Z/2)}{A\zeta^2 Z [(1-\lambda)^2 + A^2]^2 [(1+\lambda)^2 + A^2] [\cosh Z - \cos AZ]} \; *$$

$$* \; \frac{+ (e^{-Z} + e^{Z(1+\lambda)})/(1 - e^{\lambda Z})]}{}$$

(3-36)

In order to simplify equation 3-36 and to facilitate programming for digital computation, let

$$\lambda - 1 + A^2 = H,$$

$$2 - \lambda = P,$$

$$(1 - \lambda)^2 + A^2 = G, \text{ and}$$

$$\cosh Z - \cos AZ = R. \tag{3-37}$$

Then

$$I_1 = \frac{1}{3} + \frac{\coth(\lambda Z/2)}{2\lambda Z\zeta^4 G^2}$$

$$+ \frac{A\zeta^2\lambda^2[H^2 - A^2P^2 - 2PH]\sin AZ - \zeta^2\lambda^2[H^2 - A^2P^2 + 2A^2PH]\sinh Z}{4A^2ZG^2R}$$

$$+ \frac{\lambda^2[H^2 + A^2P^2]\sinh Z}{4A^2ZG^2R} - \frac{2}{\lambda\zeta^2 ZG}$$

$$+ \frac{2\lambda\zeta^2(P - H)}{ZG} + \frac{2}{\lambda Z\zeta^2 G}\left[\frac{1}{1 - e^{\lambda Z}} + \frac{1}{\lambda Z}\right]$$

$$+ \frac{2\lambda \zeta^{4} [2H - P(1 - A^{2})]}{Z^{2} G}$$

$$+ \frac{A\lambda \zeta^{2} (H - P)(e^{-Z} - \cos AZ) - \zeta^{2} \lambda (H + A^{2} P) \sin AZ}{AZGR}$$

$$+ \frac{\lambda [3A^{2} + \lambda^{2} - 1] \sin AZ - [G - 2P] A\lambda [\cos AZ \coth (\lambda Z/2) \ *}{A\zeta^{2} ZG^{2} (G + 4\lambda) R}$$

$$* \ \underline{+ (e^{-Z} + e^{Z(1 + \lambda)})/(1 - e^{\lambda Z})]}$$

(3-38)

INTRODUCTION TO THE PROGRAMMING PROGRAM

Equations 2-30 and 3-38 represent the error squared integrals for second and third order underdamped systems respectively. If the systems are critically damped,

$$\zeta = 1, \quad A = 0, \text{ and } \lim_{A \to 0} \frac{\sin AZ}{A}$$

$$= \lim_{A \to 0} Z \cos AZ = Z \tag{4-1}$$

by L' Hospital's Rule.

In this case, equation 2-30 becomes

$$I = \frac{1}{3} - \frac{4}{Z} + \frac{6}{Z^2} + \frac{5 \sinh Z - Z + 8(e^{-Z}-1)}{4Z(\cosh Z - 1)} \tag{4-2}$$

and equation 3-38 becomes

$$I_1 = \frac{1}{3} + \frac{\coth(\lambda Z/2)}{2\lambda Z H^4} + \frac{\lambda^2 H(H - 2P) Z + 2\lambda^2 P(P - H) \sinh Z}{4ZH^4 R}$$

$$- \frac{2}{\lambda Z H^2} + \frac{2\lambda(P - H)}{Z H^2} + \frac{2}{\lambda Z H^2} \left[\frac{1}{1 - e^{\lambda Z}} + \frac{1}{\lambda Z} \right]$$

$$+ \frac{2\lambda\ [2H - P]}{Z^2 H^2} + \frac{\lambda(H - P)\,(e^{-Z} - 1) - \lambda H Z}{Z H^2 R}$$

$$+ \frac{\lambda(\lambda^2 - 1)\,Z - (\lambda^2 - 3)\,\lambda\ [\coth(\lambda Z/2) + (e^{-Z} + e^{Z(1+\lambda)})/(1 - e^{\lambda Z})]}{Z H^4 R\,(H^2 + 4\lambda)}$$

(4-3)

If the systems are overdamped, $\zeta > 1$ and A is imaginary. Then

$$A = j\,\frac{\sqrt{\zeta^2 - 1}}{\zeta} = jA_0\ ,$$

$$\cos jA_0 Z = \cosh A_0 Z, \text{ and}$$

$$-j \sin jA_0 Z = \sinh A_0 Z.$$

(4-4)

For the second order overdamped case,

$$I = \frac{1}{3} - \frac{4\zeta^2}{Z} + \frac{2\zeta^2\,(4\zeta^2 - 1)}{Z^2}$$

$$+ \frac{(1 + 4\zeta^2)\sinh Z + (3 - 4\zeta^2)\,(\sinh A_0 Z)/A_0 + 8\zeta^2\,(e^{-Z} - \cosh A_0 Z)}{4Z\,(\cosh Z - \cosh A_0 Z)}$$

(4-5)

and for the third order overdamped case

$$I_1 = \frac{1}{3} + \frac{\coth(\lambda Z/2)}{2\lambda Z\zeta^4 G^2}$$

$$+ \frac{A\lambda^2\zeta^2 [H^2 + A^2P^2 - 2PH] \sinh AZ + \zeta^2\lambda^2 [H^2 + A^2P^2 - 2A^2PH] \sinh Z}{4ZRA^2G^2}$$

$$+ \frac{\lambda^2 [A^2P^2 - H^2] \sinh Z}{4A^2ZRG^2} - \frac{2}{\lambda\zeta^2 ZG} + \frac{2\lambda\zeta^2 (P - H)}{ZG}$$

$$+ \frac{2}{\lambda ZG\zeta^2} \left[\frac{1}{\lambda Z} + \frac{1}{1 - e^{\lambda Z}}\right] + \frac{2\lambda\zeta^4 [2H - P(1 + A^2)]}{Z^2G}$$

$$+ \frac{A\lambda\zeta^2 [H - P] [e^{-Z} - \cosh AZ] - \zeta^2\lambda [H - A^2P] \sinh AZ}{AZGR}$$

$$+ \frac{\lambda [\lambda^2 - 3A^2 - 1] \sinh AZ - [\lambda^2 - A^2 - 3] A\lambda [\cosh AZ \coth(\lambda Z/2)}{A\zeta^2 ZRG^2 (G + 4E)} *$$

$$* \underline{+ (e^{-Z} + e^{Z(1+\lambda)}) / (1 - e^{\lambda Z})]} \; .$$

(4-6)

Consider first the second order system. The problem is to find a d.c. loop gain-frequency function which minimizes I,

or to find an equivalent relationship of the form $\zeta^2 = f(Z)$ which minimizes I. This can be done by setting the derivative of I with respect Z equal to zero and solving the resulting expression explicitly for $\zeta^2 = f(Z)$. Even the simplest of the error squared integrals is too complex to handle analytically in this manner.

Because of the complex structure of the I functions, digital computer techniques were used to plot and locate the minimums of these functions. All programs were written in the language of Burrough's Algebraic Compiler. A discussion of the programming techniques used is taken up on the following pages. A listing of all programs used appears in the appendix.

PROGRAMMING FOR THE SECOND ORDER ERROR INTEGRAL

The evaluation of I for several values of the parameter ζ was simply a matter of substituting values into the derived functions I for $\zeta > 1$, $\zeta = 1$, $\zeta < 1$. Rough plots of the I family of curves were made first using slide rule calculations. These plots were used to select points for the input arrays. There were unexpected irregularities in some of the curves, especially those of very low ζ which made it necessary to rerun the three programs for additional points. Resulting curves are shown in Figures 5 and 6.

The second phase of the problem is that of accurately locating the true absolute minimum for each curve. An analytic expression for the derivative of I with respect to Z is required in the minimization procedure used. The three forms of I' (Z) are listed below for reference.

Underdamped $I'(Z) = \dfrac{4\zeta^2}{Z^2} - \dfrac{4\zeta^2 (4\zeta^2 - 1)}{Z^3}$

$+ \{ [(1 + 4\zeta^2) \cosh Z + (3-4\zeta^2) \cos AZ + 8\zeta^2 (A \sin AZ - e^{-Z})]$

$$Z\,[\cosh Z - \cos AZ] - [(1 + 4\zeta^2)\sinh Z$$

$$+ (3 - 4\zeta^2)\,\frac{\sin AZ}{A} + 8\zeta^2\,(e^{-Z} - \cos AZ)\,]\;[\cosh Z$$

$$+ Z \sinh Z - \cos AZ + AZ \sin AZ]\Big\}\;/4Z^2\,[\cosh Z - \cos AZ\,]^2$$

(4-7)

$$\text{Overdamped } I'(Z) = \frac{4\zeta^2}{Z^2} - \frac{4\zeta^2\,(4\zeta^2 - 1)}{Z^3}$$

$$+ \Big\{[(1 + 4\zeta^2)\cosh Z + (3 - 4\zeta^2)\cosh A_0 Z - 8\zeta^2\,(A_0 \sin A_0 Z$$

$$-e^{-Z})\,]\,Z\;[\cosh Z - \cosh A_0 Z] - [(1 + 4\zeta^2)\sinh Z$$

$$+ (3 - 4\zeta^2)\,\frac{\sinh A_0 Z}{A_0} + 8\zeta^2\,(e^{-Z} - \cosh A_0 Z)]\;[\cosh Z$$

$$+ Z \sinh Z - \cosh A_0 Z - A_0 Z \sinh A_0 Z]\Big\}\;/4Z^2\;[\cosh Z - \cosh A_0 Z]^2$$

(4-8)

$$\text{Critically damped } I'(Z) = \frac{4}{Z^2} - \frac{12}{Z^3}$$

$$+\Big\{[5 \cosh Z - 1 - 8e^{-Z}]\;Z\,[\cosh Z - 1] - [5 \sinh Z$$

$$- Z + 8(e^{-Z}-1)] \; [\cosh Z + Z \sinh Z - 1] \Big\} / 4Z^2[\cosh Z - 1]^2 \qquad (4\text{-}9)$$

A plan for accelerating the convergence of iterative processes described by J. H. Wegstein in Communications of the Association for Computing Machinery, Volume 1, number 6, page 9, of June, 1958, was used to find the roots of the family of curves $I'(Z) = 0$. The procedure is explained below.

1. Write $I'(Z) = 0$ in the form

   $Z = Z + gI'(Z) = F(Z)$, where $g \neq 0$ is a constant.

2. Choose a Z_0 (refer to Figure 7, page 59)
3. Let $Z_1 = F(Z_0)$
4. Let $Z_2 = F(Z_1)$

Instead of using Z_2 to continue the process, find the intersection of the secant joining Z_0, $F(Z_0)$ and Z_1, $F(Z_1)$ with the line $y = Z$ and call this value $\overline{Z}_2$. For each new Z, follow this same procedure. Each $\overline{Z}_{n+1}$ is obtained from the intersection of $y = Z$ and the secant joining $\overline{Z}_n$, $F(\overline{Z}_n)$ and $\overline{Z}_{n-1}$, $F(\overline{Z}_{n-1})$. Note from the graph that each $F(\overline{Z}_n) = Z_{n+1}$. From the geometry of the graph, there are two relationships that are of use.

$$\overline{Z}_{n+1} = \frac{Z_{n+1}\overline{Z}_{n-1} - Z_n\overline{Z}_n}{Z_{n+1}\overline{Z}_{n+1} - Z_n - \overline{Z}_n}$$

(4-10)

and

$$Z_{n+1} = F(\overline{Z}_n)$$

(4-11)

A flow chart for the procedure appears in Figure 8.

Several tries were required before the process would work satisfactorily. The problem was to find the right combination of g (the arbitrary constant multiplier of I' (Z) and h (for the convergence test $|\overline{Z}_{n+1} - \overline{Z}_n| < h$). Too large a g caused the hyperbolic functions to go out of range and too small a g caused overflow in the calculation of $\overline{Z}_{n+1}$. h also had rather narrow limits. Results were not reliable unless h was kept near some very small value, yet when h was made too small, Z oscillated without converging. It is also evident from the plots of I(Z) that it was necessary to have a good idea of where the absolute minimums were located since there are several roots to the derivative equations in some cases. Results of the minimizations are shown in Figure 9. The surprising fact is that

$\zeta^2 = f(Z)$ is a linear function, therefore no curve fitting problem exists! The equation of the minimizing function is:

$$\zeta^2 = 0.256Z. \tag{4-12}$$

PROGRAMMING FOR THE THIRD ORDER ERROR INTEGRAL

The third order case is complicated by an additional parameter λ. Evaluation of I_1 was performed by using the same procedure that was used in the second order case except that data was read into the program by using the FOR statement rather than by an input array. Several sets of points were needed; one set for each family of curves, since the minimums form a surface rather than a plane curve. Plots of I_1 (Z) appear in Figures 10, 11, and 12.

An examination of the third order error squared integral will reveal why it was impractical to use Wegstein's method, which requires the derivative of I_1 (Z), for locating the minimums of each function. They were found instead by constructing a net from the plots of I_1 (Z) in the neighborhood of each minimum. Spacing of the net was decreased until 3 points were determined: Z_0, $Z_0 + \Delta$, and $Z_0 - \Delta$ such that I_1 (Z_0) was less than either I_1 $(Z_0 + \Delta)$ or I_1 $(Z_0 - \Delta)$ where Δ is the tolerable error in Z. Results of this analysis are shown in the family of straight lines in Figure 13. Each line represents a minimum function $\zeta^2 = f(Z)$

for a particular value of λ. These lines are parallel, therefore, the minimizing surface can be expressed as

$$\zeta^2 = mZ + f(b) \quad ,$$

where b is the ordinant of the ζ^2 axis intercept for each line, and m is the slope of the family of straight lines.

By determining a functional relationship $b = f(\lambda)$, an analytical expression for $\zeta^2 = f(Z, \lambda)$ can be written. A plot of $b = f(\lambda)$ is shown in Figure 14. The resulting curve forms one leg of an equilateral hyperbola. Its equation is:

$$\lambda b = .515 \quad , \tag{4-13}$$

therefore

$$\zeta^2 = .256Z + \frac{.515}{\lambda} \tag{4-14}$$

is the required minimizing function for the third order system.

There is a very slight scattering of points in Figure 13. This can be attributed to the complex function which was being evaluated. When such a large number of computations involving

extremely large variations in magnitude are required to plot a single point, small errors are bound to occur.

CONCLUSIONS

$$y = t/T \tag{5-1}$$

$$y_1 = t/T + b_1 \tag{5-2}$$

Mathematically, a delay in the system input pulse sequence is equivalent to shifting the desired output $y = t/T$ as shown in Figure 4. The analysis of this paper was based on a linear desired output (equation 5-1). Optimizing functions were derived based upon minimization of

$$\frac{1}{T}\int_{KT}^{(K+1)T} [\theta(t) - t/T]^2 \, dt. \tag{5-3}$$

It can be seen from plots of this integral that in all cases it approaches zero for some T.

If a similar study is made based on equation 5-2 for a delayed pulse sequence, the integral to be minimized becomes:

$$I = \frac{1}{T}\int_{KT}^{(K+1)T} [\theta(t) - (t/T + b_1)]^2 \, dt$$

$$= \frac{1}{T}\int_{KT}^{(K+1)T} [\theta(t) - t/T]^2 \, dt - \frac{2b_1}{T}\int_{KT}^{(K+1)T} [\theta(t) - t/T] \, dt + \frac{b_1^{\,2}}{T}\int_{KT}^{(K+1)T} dt$$

(5-4)

The first member of the right hand side of equation 5-4 has been plotted for second and third order systems in sections 2 and 3 of this paper. The value of the third integral is $b_1^{\,2}$. The second integral can be handled with the aid of equations 2-8, 2-13, and 3-9. In the steady-state second order case:

$$- \frac{2b_1}{T}\int_{KT}^{(K+1)T} [\theta(t) - t/T] \, dt = - \frac{2b_1}{T}\int_0^T \left\{1 - \frac{x}{T} + \right.$$

$$\left. + \frac{1}{2j\beta} \left[\frac{(\alpha + j\beta)e^{-(\alpha - j\beta)x}}{e^{-(\alpha - j\beta)T} - 1} - \frac{(\alpha - j\beta)e^{-(\alpha + j\beta)x}}{e^{-(\alpha + j\beta)T} - 1} \right] \right\} dx$$

$$= - \frac{2b_1}{T} \left[T - \frac{T^2}{2T} + \frac{\alpha + j\beta}{2j\beta\,[-(\alpha - j\beta)]} - \frac{\alpha - j\beta}{2j\beta\,[-(\alpha + j\beta)]} \right]$$

$$= - b_1 - \frac{2b_1}{2jT\beta\beta_0^{\,2}} \quad [(\alpha - j\beta)^2 - (\alpha + j\beta)^2]$$

$$= - b_1 + \frac{4b_1\alpha}{\beta_0^{\,2}T}$$

(5-5)

If the terms added to the integral I by consideration of a pulse delay are denoted by I_s, then for the second order case

$$I_s = b_1^{\,2} - b_1 + \frac{4a\,b_1}{\beta_0^{\,2}T} \,. \tag{5-6}$$

In terms of ζ and Z, equation 5-6 becomes:

$$I_s = b_1^{\,2} - b_1 + 4b_1 \frac{Z}{Z^2(1 + A^2)} = b_1^{\,2} + \frac{4b_1\zeta^2}{Z} - b_1 \,. \tag{5-7}$$

The third order case can be handled similarly.

$$- \frac{2b_1}{T}\int_{KT}^{(K+1)T} [\theta(t) - t/T]\, dt = - \frac{2b_1}{T}\int_0^T \left\{ 1 - \frac{x}{T} + \frac{\beta_0^{\,2}}{\gamma_0^{\,2}} \frac{e^{-\gamma x}}{(e^{-\gamma T}-1)} \right.$$

$$\left. + \frac{\gamma}{2j\beta\gamma_0^{\,2}} \left[\frac{(M - jN)e^{-(a - j\beta)x}}{e^{-(a - j\beta)T}-1} - \frac{(M + jN)e^{-(a + j\beta)x}}{e^{-(a + j\beta)T}-1} \right] \right\} dx$$

$$= - \frac{2b_1}{T} \left[T - \frac{T^2}{2T} + \frac{\beta_0^{\,2}}{\gamma_0^{\,2}} \cdot \left(- \frac{1}{\gamma} \right) + \frac{\gamma}{2j\beta\gamma_0^{\,2}} \left[\frac{M - jN}{-(a - j\beta)} - \frac{M + jN}{-(a + j\beta)} \right] \right]$$

$$= -b_1 + \frac{2b_1\beta_0^2}{\gamma\gamma_0^2 T} + \frac{2b_1\gamma(2)}{2jT\beta\gamma_0^2} \quad \mathrm{Im}\left[\frac{M - jN}{\alpha - j\beta} - \frac{M + jN}{\alpha + j\beta}\right]$$

(5-8)

$$\frac{1}{\beta_0^2} \quad \mathrm{Im}\,[(M - jN)(\alpha + j\beta) - (M + jN)(\alpha - j\beta)]$$

$$= \frac{2}{\beta_0^2}\,[\beta M - \alpha N]$$

(5-9)

Substituting equation 5-9 into 5-8, I_s for the third order case can be written

$$I_{1s} = b_1^2 - b_1 + \frac{2b_1\beta_0^2}{\gamma\gamma_0^2 T} + \frac{4b_1\gamma}{\beta T\beta_0^2\,\gamma_0^2} \quad [\beta M - \alpha N]$$

(5-10)

In terms of ζ , Z, and λ, equation 5-10 becomes:

$$I_{1s} = b_1^2 - b_1 + \frac{2b_1 Z^2(1 + A^2)}{\lambda Z \cdot Z^2\,[(1 - \lambda)^2 + A^2]}$$

$$+ \frac{4b_1\lambda Z\,[AZ \cdot Z^2\,(\lambda - 1 + A^2) - Z \cdot AZ^2\,(2 - \lambda)]}{AZ \cdot Z^2\,(1 + A^2)\,Z^2\,[(1 - \lambda)^2 + A^2]}$$

$$= b_1^2 - b_1 + \frac{2b_1}{\lambda\zeta^2 Z\,[1/\zeta^2 + \lambda^2 - 2\lambda]} + \frac{4b_1\lambda\zeta^2\,[\frac{1}{\zeta^2} + 2\lambda - 4]}{Z\,[\frac{1}{\zeta^2} + \lambda^2 - 2\lambda]} \tag{5-11}$$

Plots for equation 5-7 are shown in Figures 15 through 20 for several values of ζ and for $b_1 = 0.25,\ 0.50,\ 0.75$. A summary of the results of varying b_1 are shown in Figures 21 and 22. Increasing b_1 beyond 0.25 makes it impossible to obtain a zero value for I. This effect is even more pronounced as ζ is increased. Even for $\zeta = 0.5$, no clearly defined minimum exists for $b_1 \geq 0.5$. In all cases where it is possible to obtain high accuracy (I = 0), the optimum frequencies are lowered considerably by the pulse delay. Pulse input delays do, however, flatten the I curves in the vicinity of the minimums so that when I can be made to approach zero, gain-frequency selection is less critical when delay is involved.

Equation 5-11 is plotted in Figure 23 for $\lambda = 10$, $\zeta = 0.1$ and $b_1 = 0.25,\ 0.5,\ 0.75$. These curves should be added to the $\zeta = 0.1$ curve of Figure 11. A casual inspection will show that considerable error is involved for even the $b_1 = 0.25$ curve. In-

creasing either ζ or b_1 makes the situation worse so that in the third order system not even $b_1 = .25$ can be tolerated.

Plots of the error squared integral functions show that a linear system output can be generated to any desired degree of accuracy by selecting system parameters according to equations 4-12 and 4-14. They also indicate relative errors which will result from not selecting parameters correctly. The slope of all curves increases very rapidly to the left of the minimums (higher frequencies) and these minimums are much more sharply defined for curves of small damping ratio.

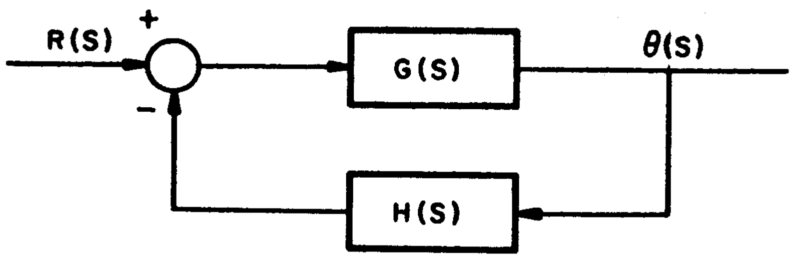

FIG. 1 – A GENERAL LINEAR FEEDBACK SYSTEM

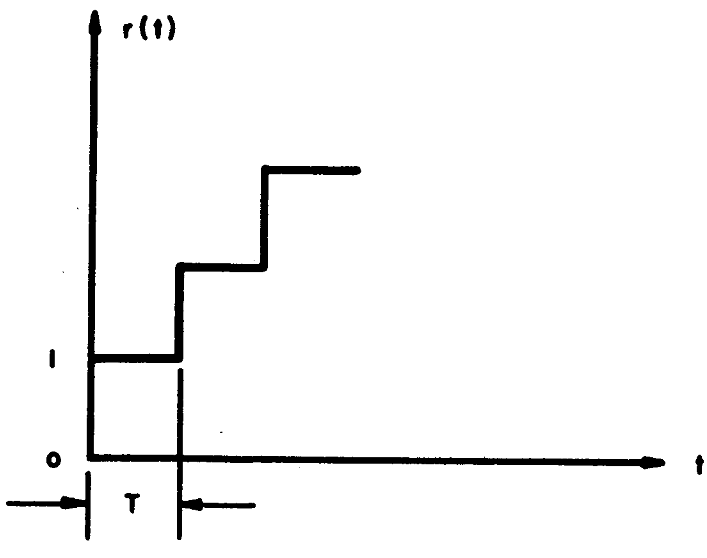

FIG. 2 - REFERENCE INPUT FUNCTION r(t)

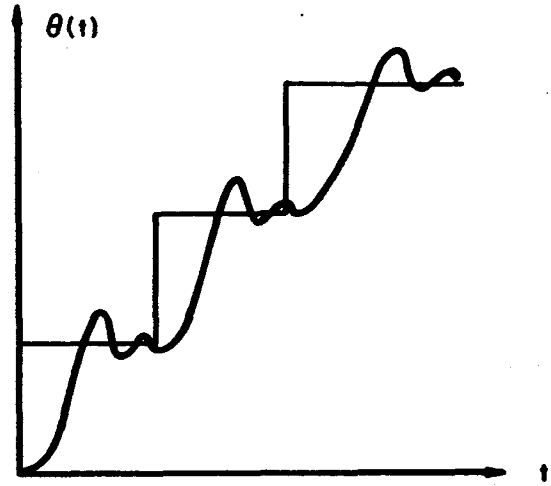

FIG. 3 - OUTPUT θ(t) SHOWING VARIABLE INITIAL CONDITIONS

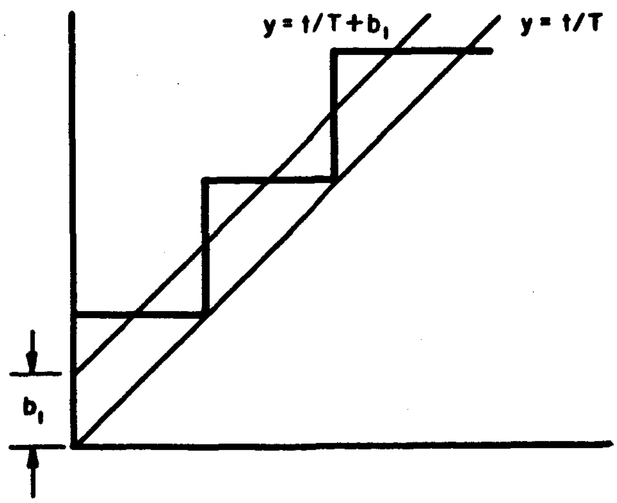

FIG. 4 - y(t) FOR DELAY IN THE INPUT PULSE SEQUENCE

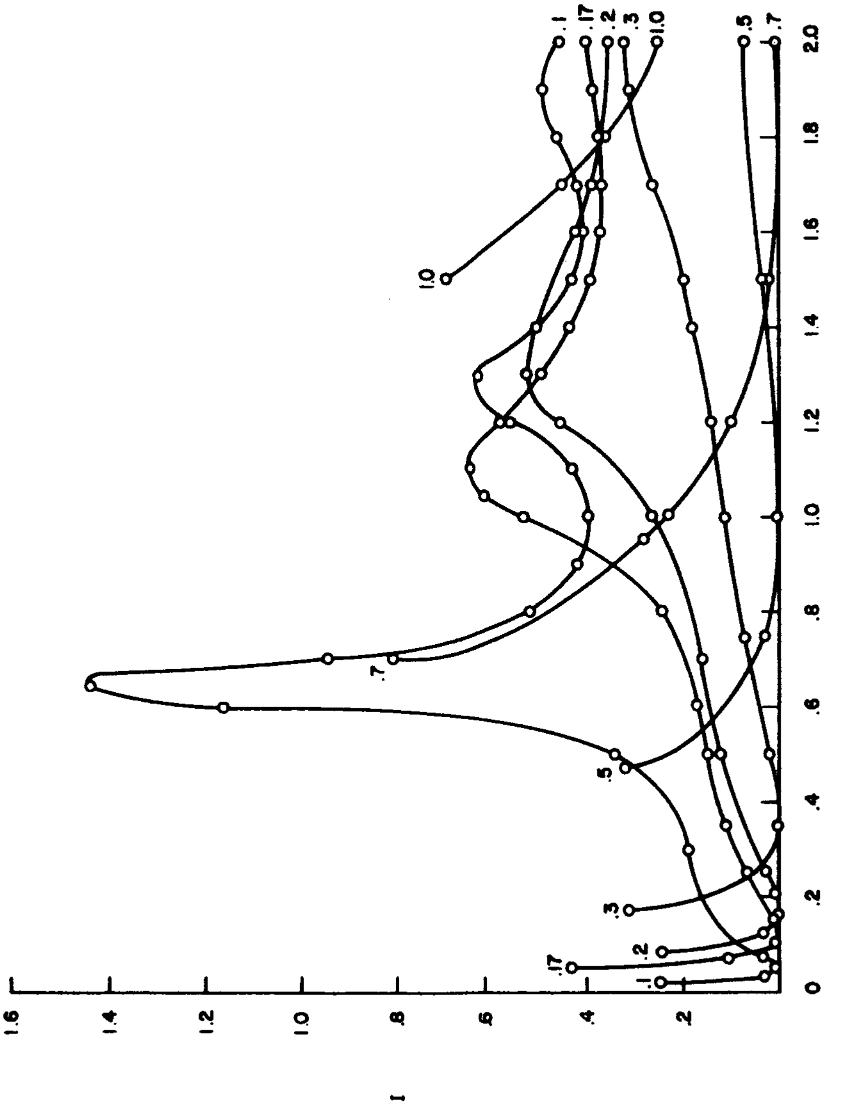

FIG. 5 - I = f(Z), PARAMETER ζ

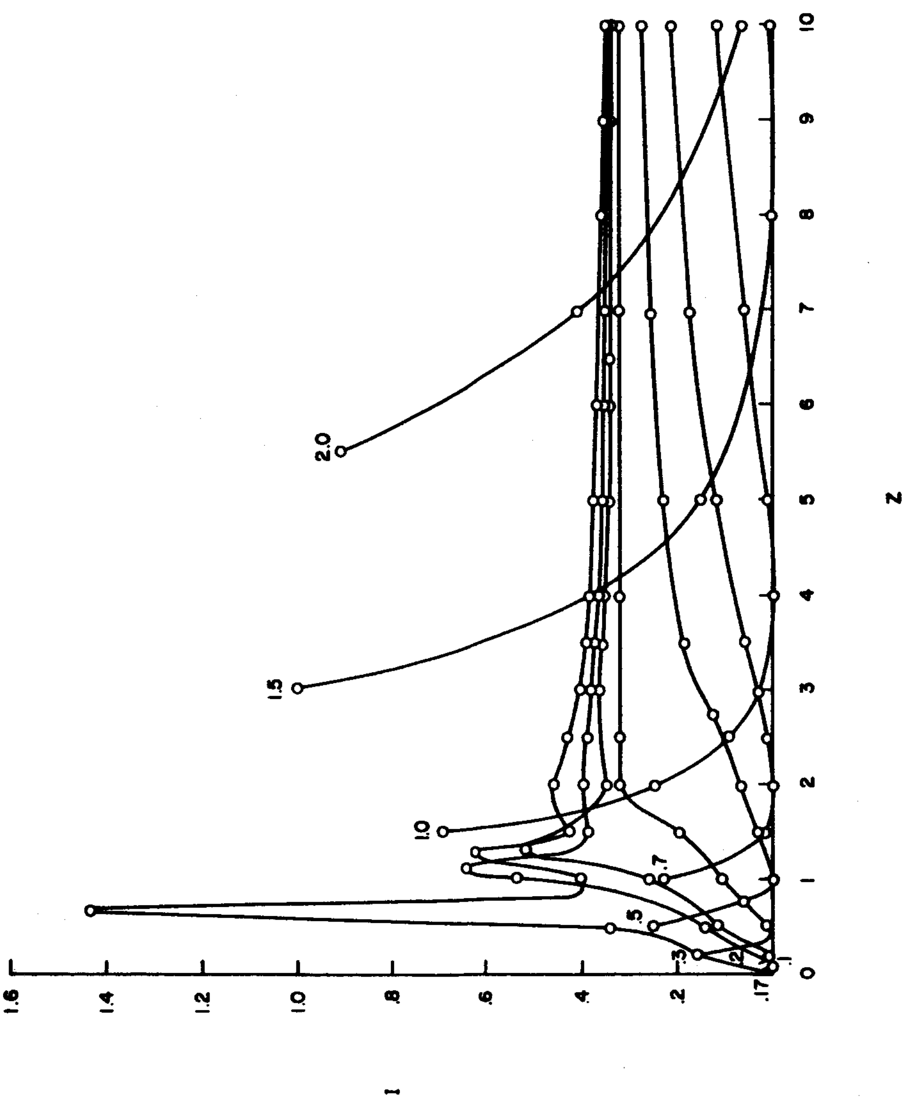

FIG. 6 – I = f(Z), PARAMETER ζ

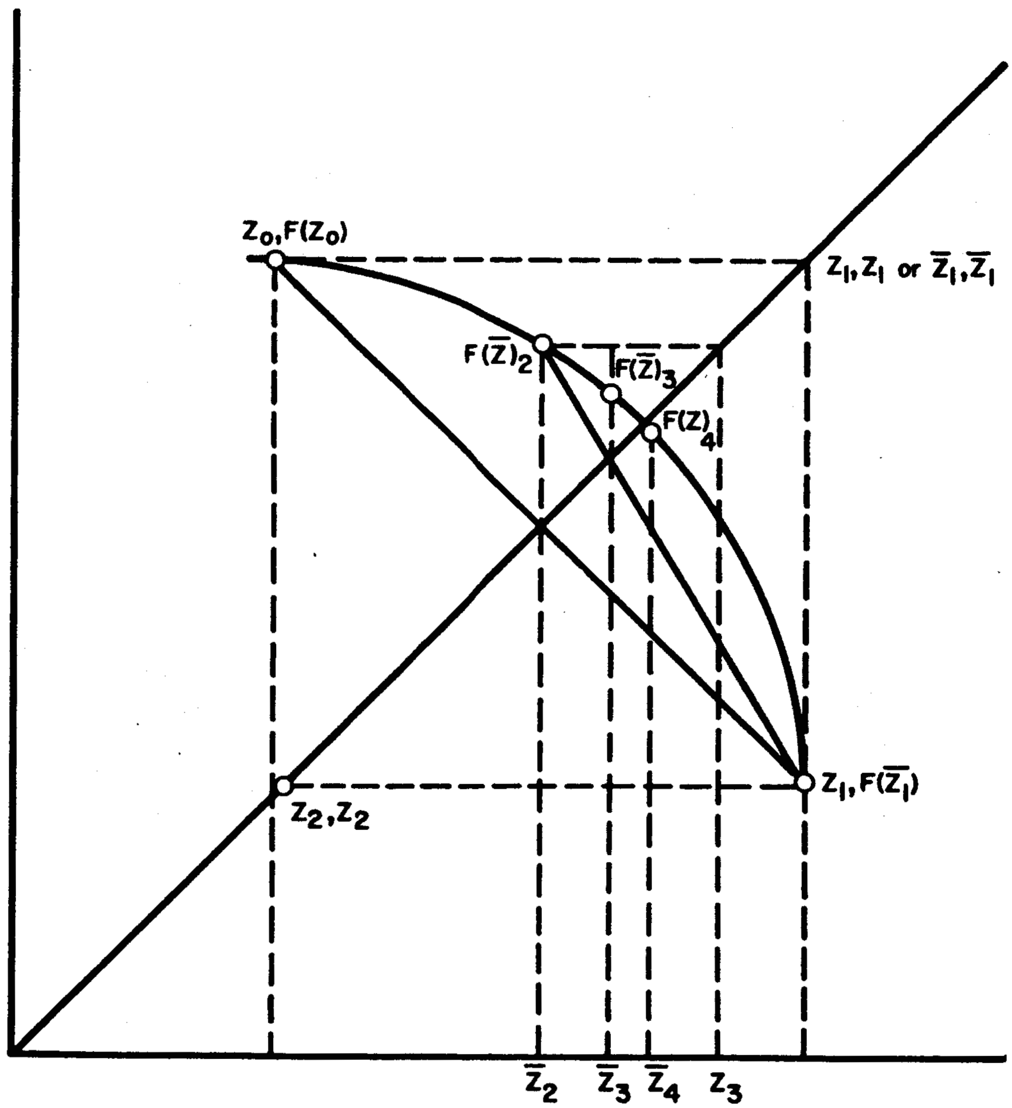

FIG. 7 – WEGSTEIN'S ITERATION PROCESS

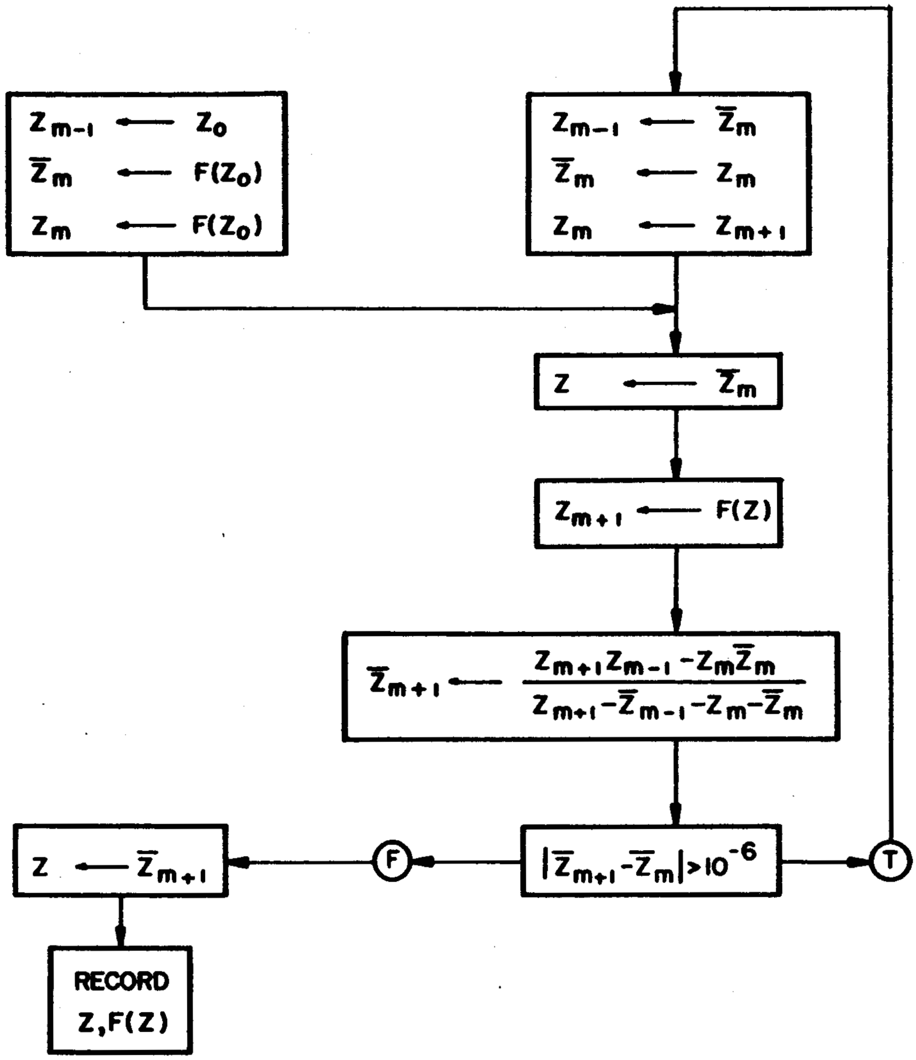

FIG. 8 – FLOW CHART FOR WEGSTEIN'S INERATIVE PROCESS

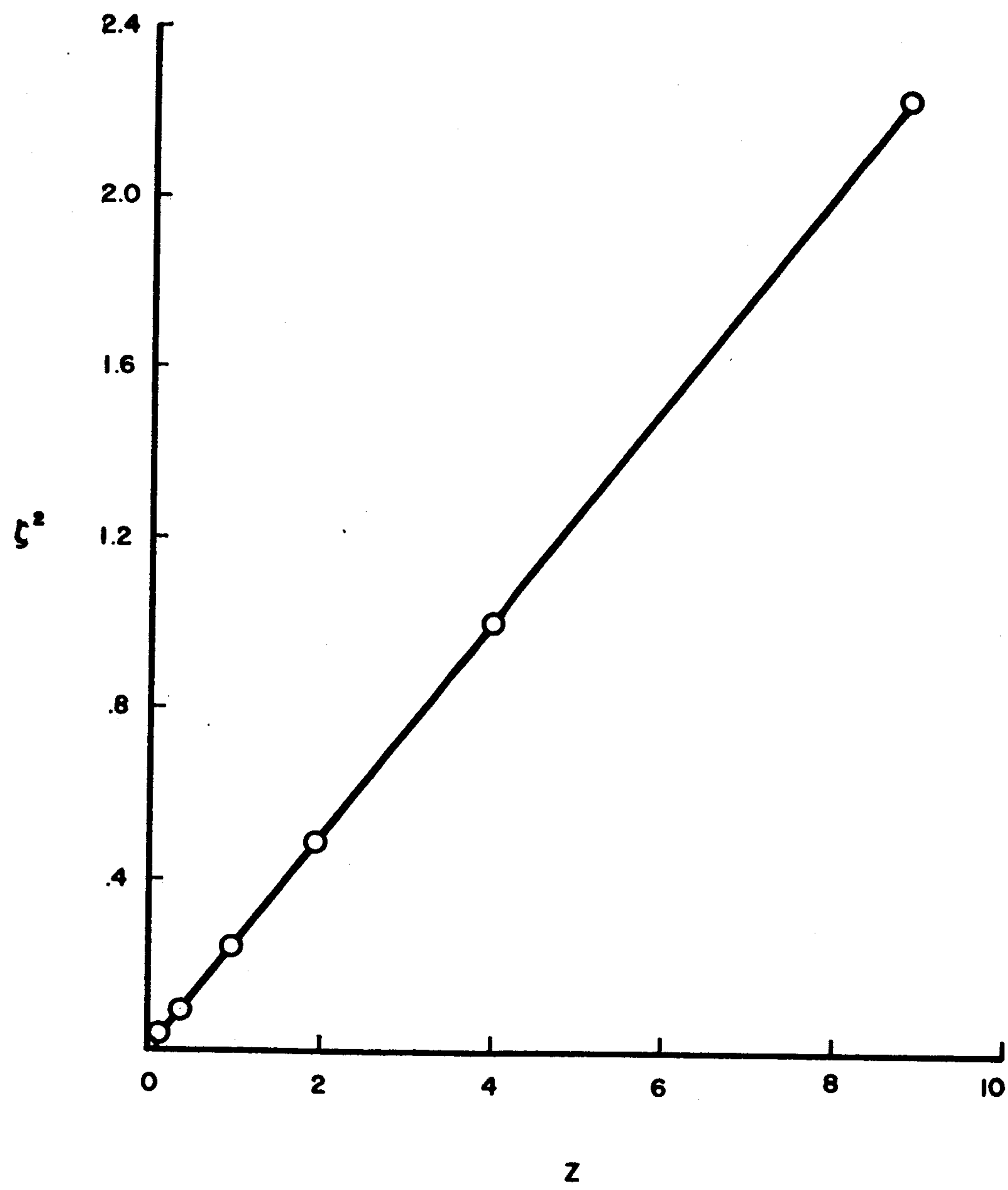

FIG. 9 — $\zeta^2 = f(Z\ MIN)$

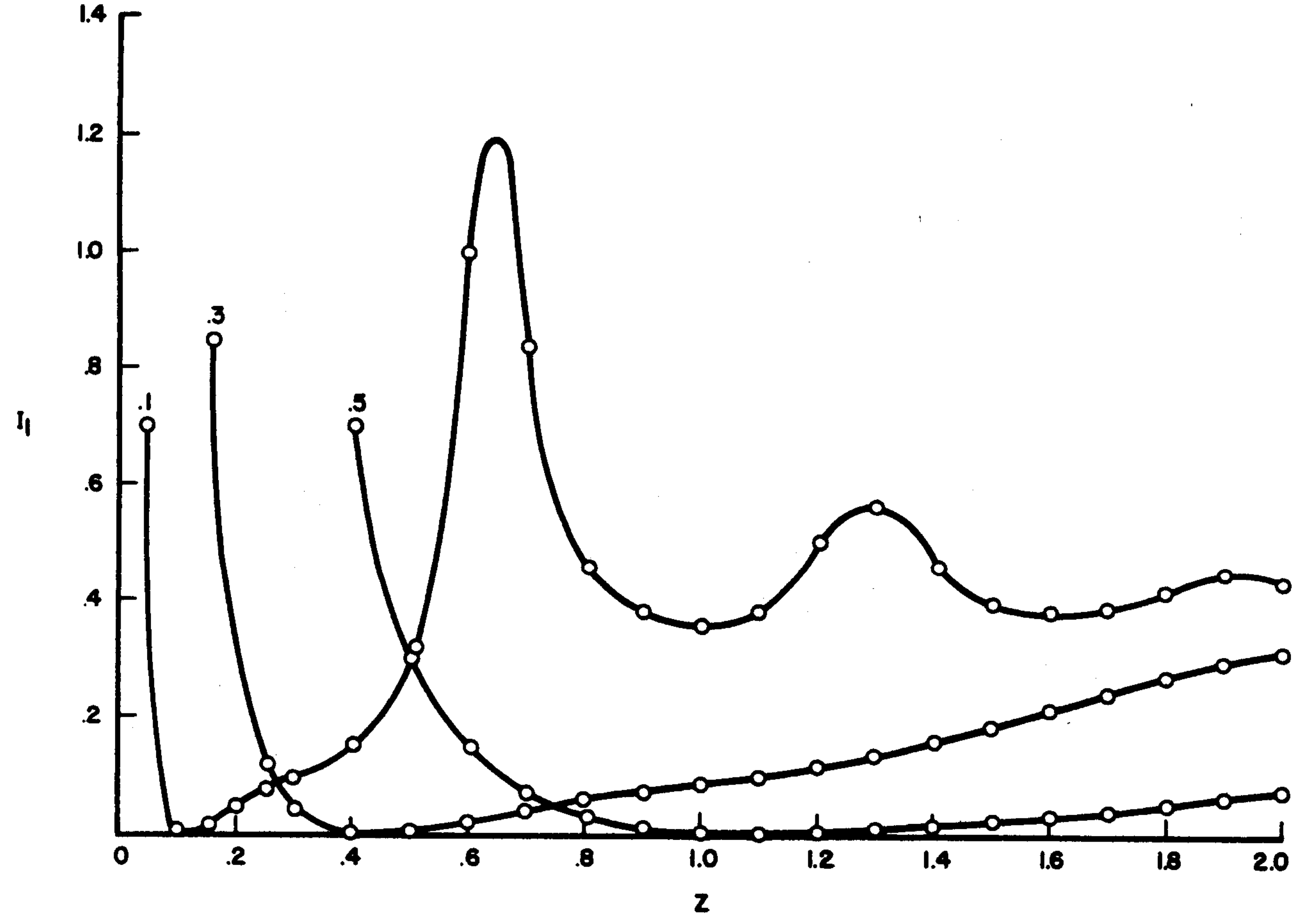

FIG. 10 – $I = f(Z)$, $\lambda = 30$, PARAMETER ζ

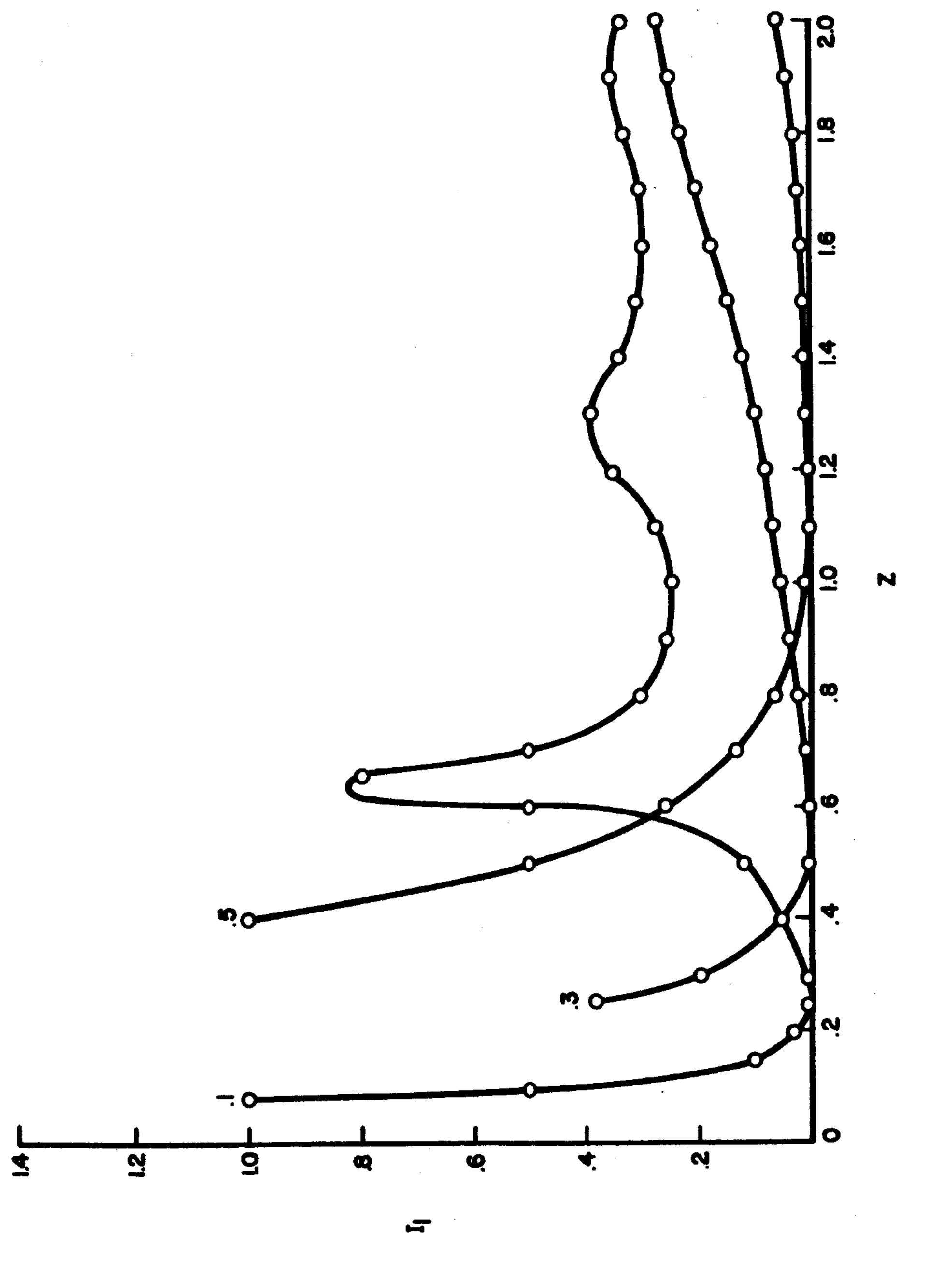

FIG. 11 — $I_1 = f(Z)$, $\lambda = 10$, PARAMETER ζ

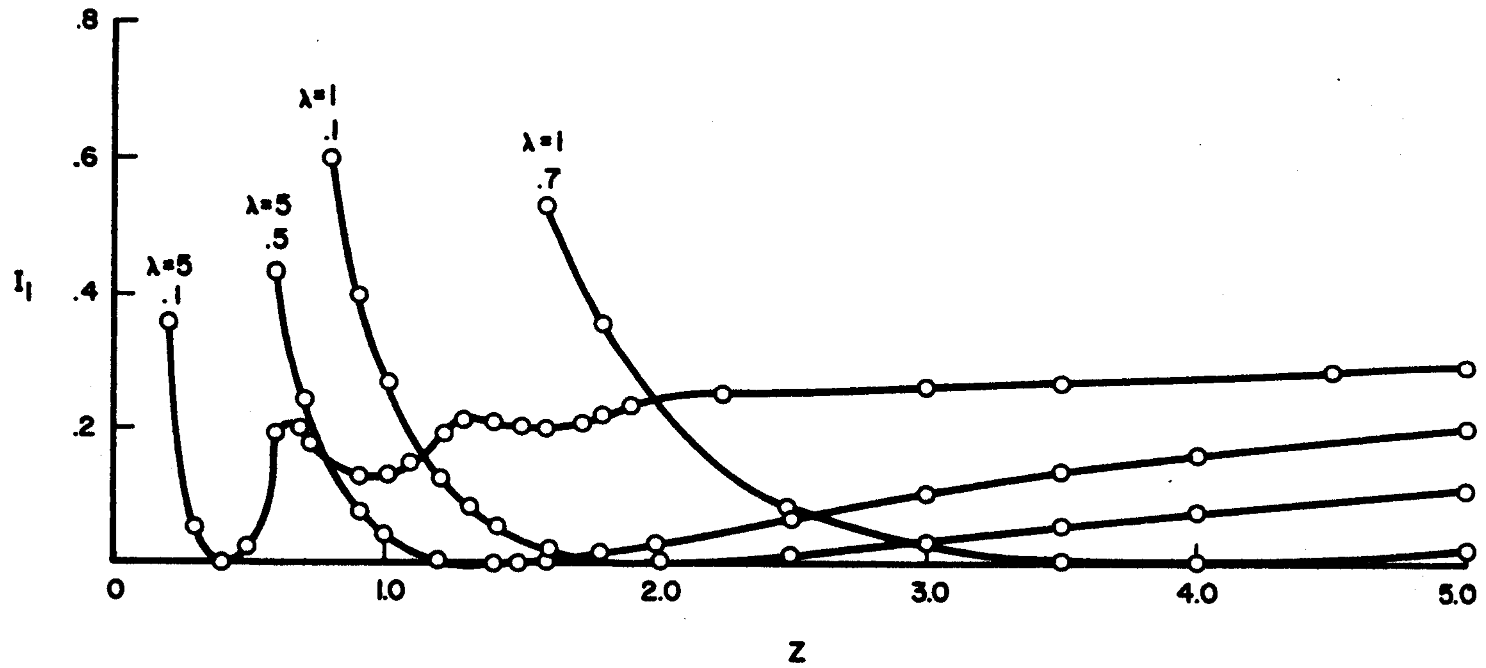

FIG. 12 – $I_3 = f(Z)$, $\lambda = 1$ & 5, PARAMETER ζ

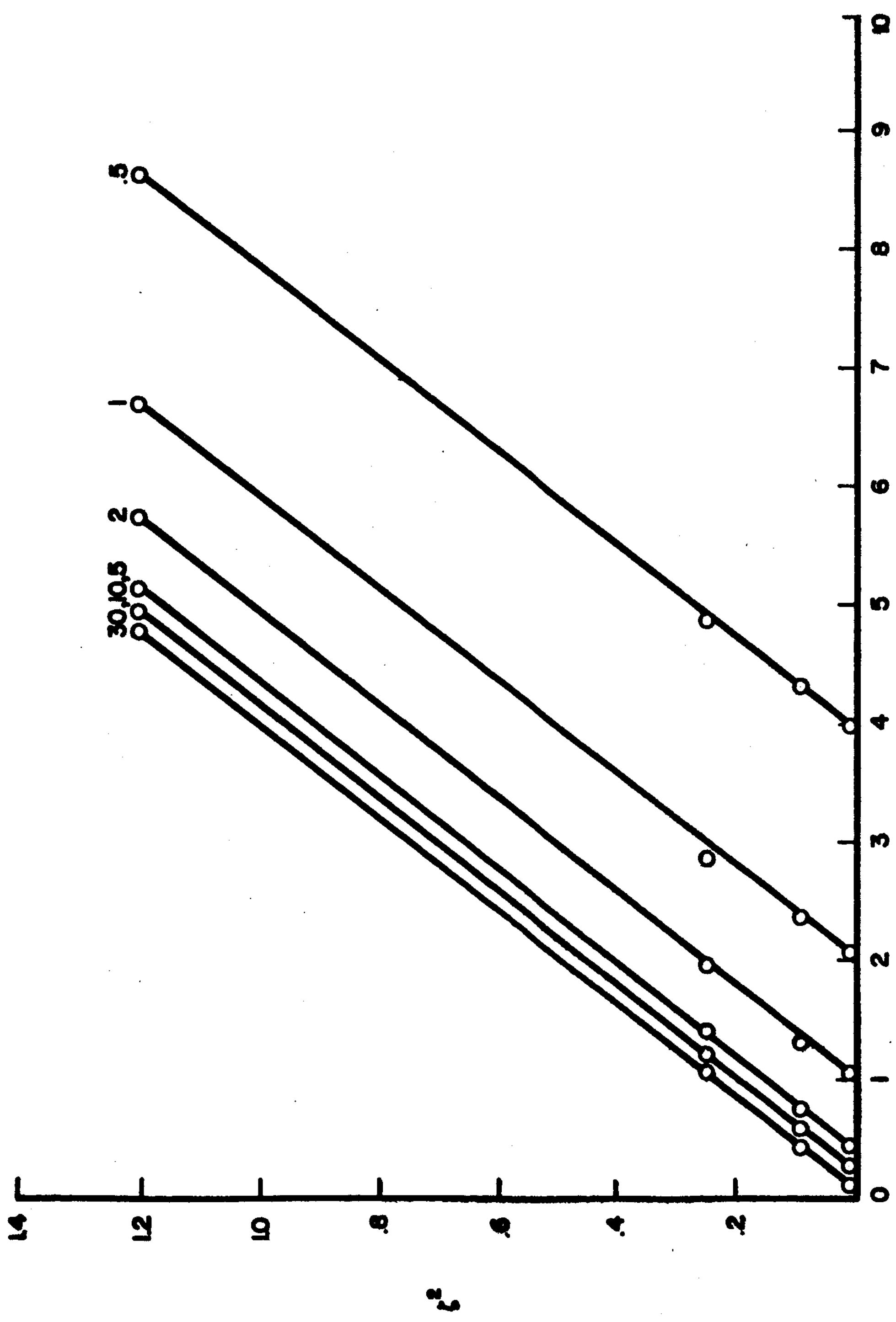

FIG. 13 — $\zeta^2 = f(Z\,\mathrm{MIN})$, PARAMETER λ

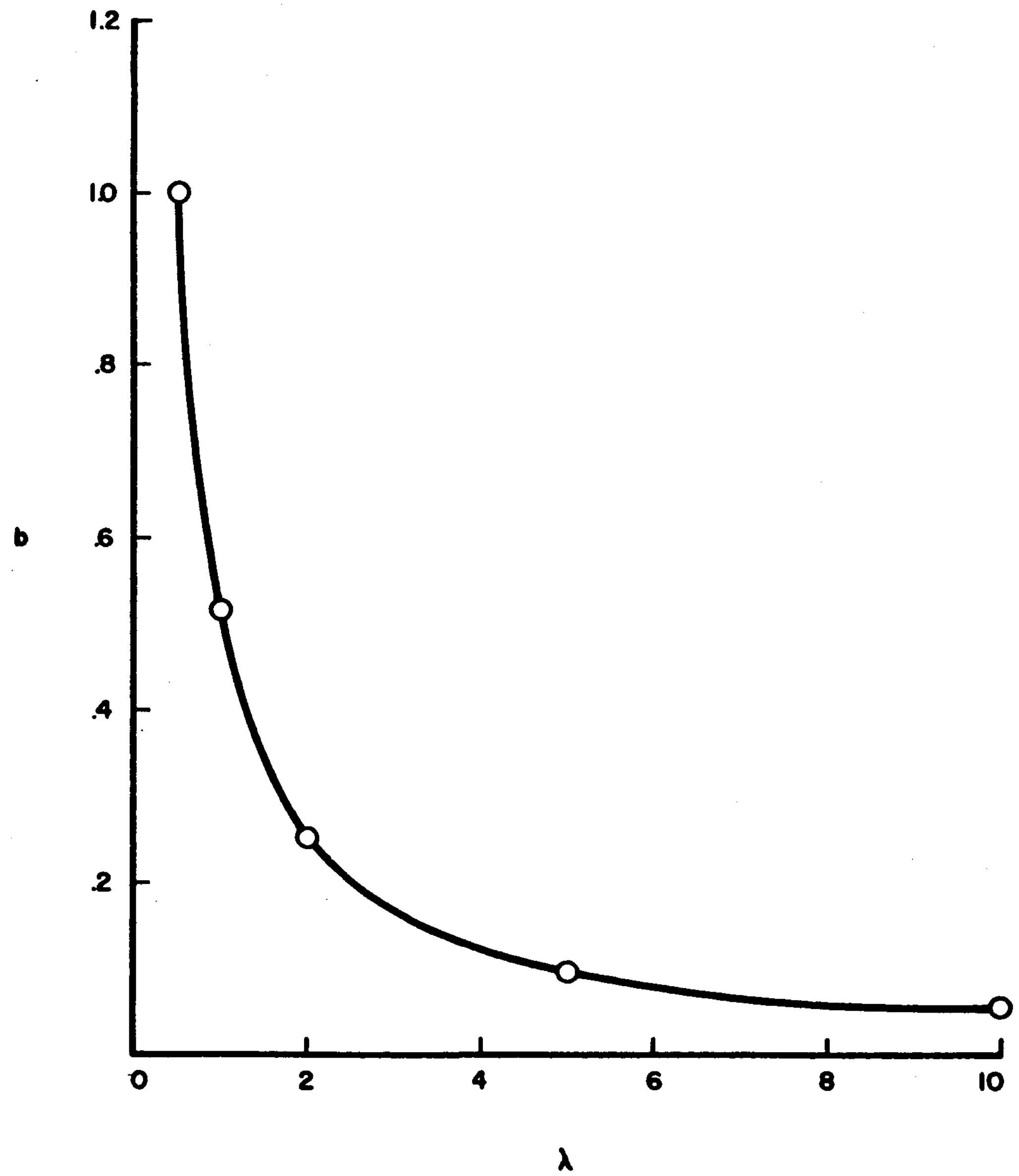

FIG. 14 — $b = f(\lambda)$

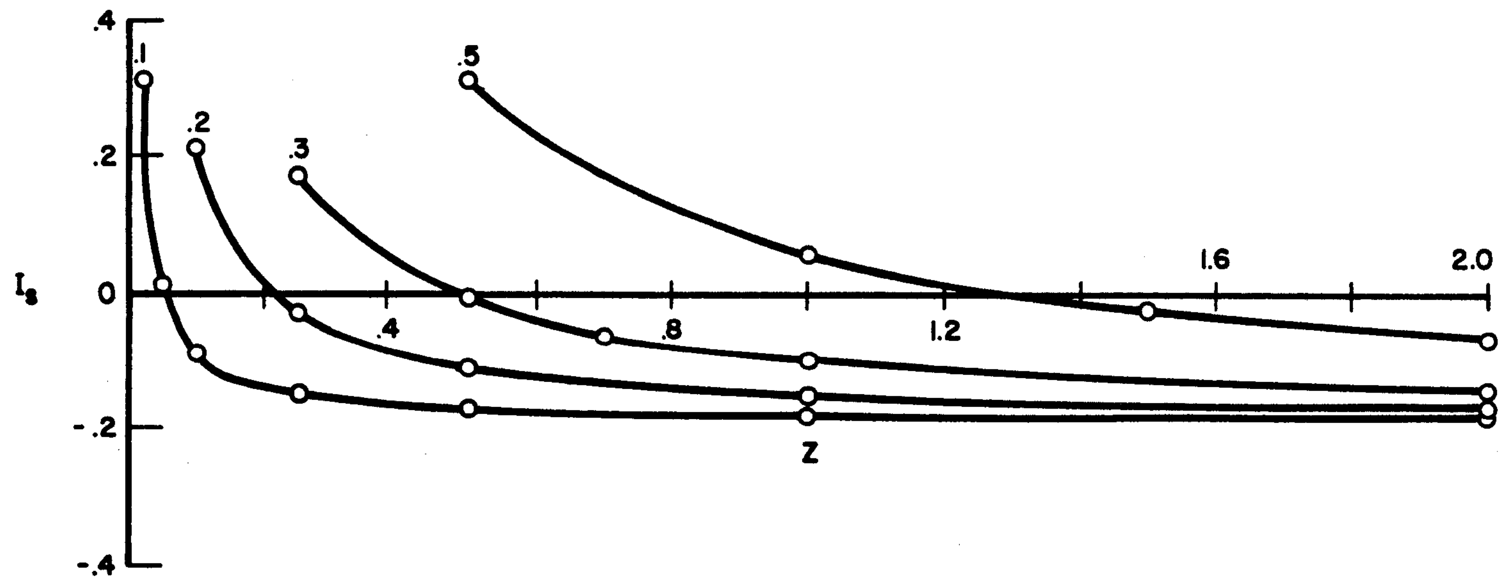

FIG. 15 – $I_s = f(Z)$, $b_1 = 0.25$, PARAMETER ζ

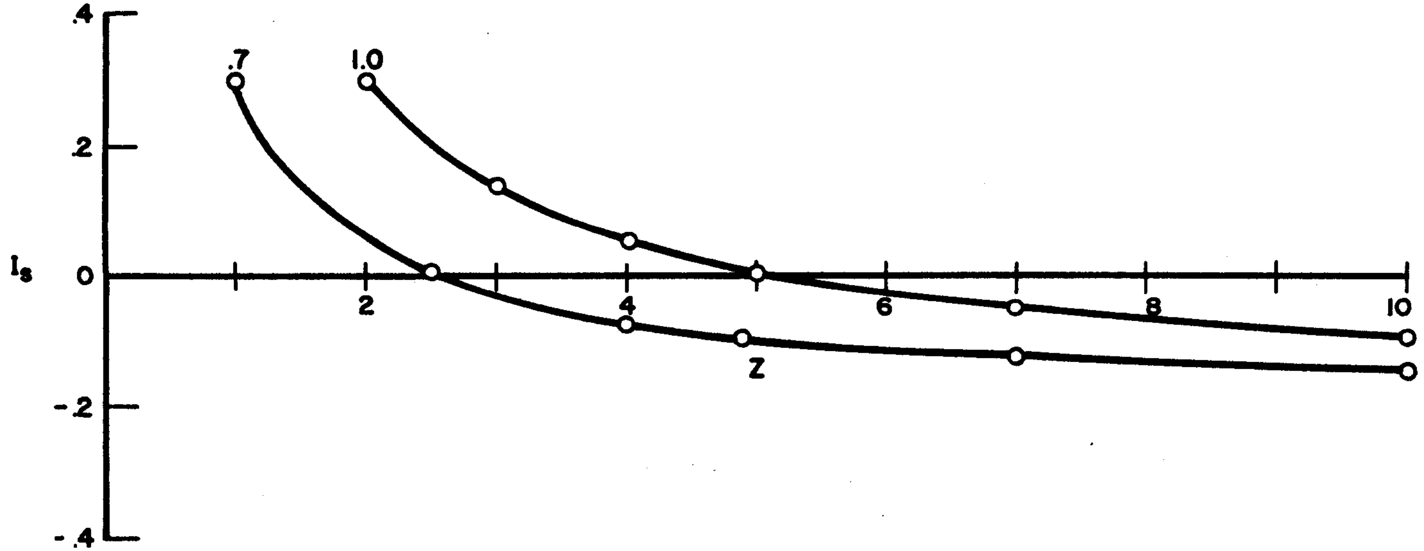

FIG. 16 — $I_s = f(Z)$, $b_1 = 0.25$, PARAMETER ζ

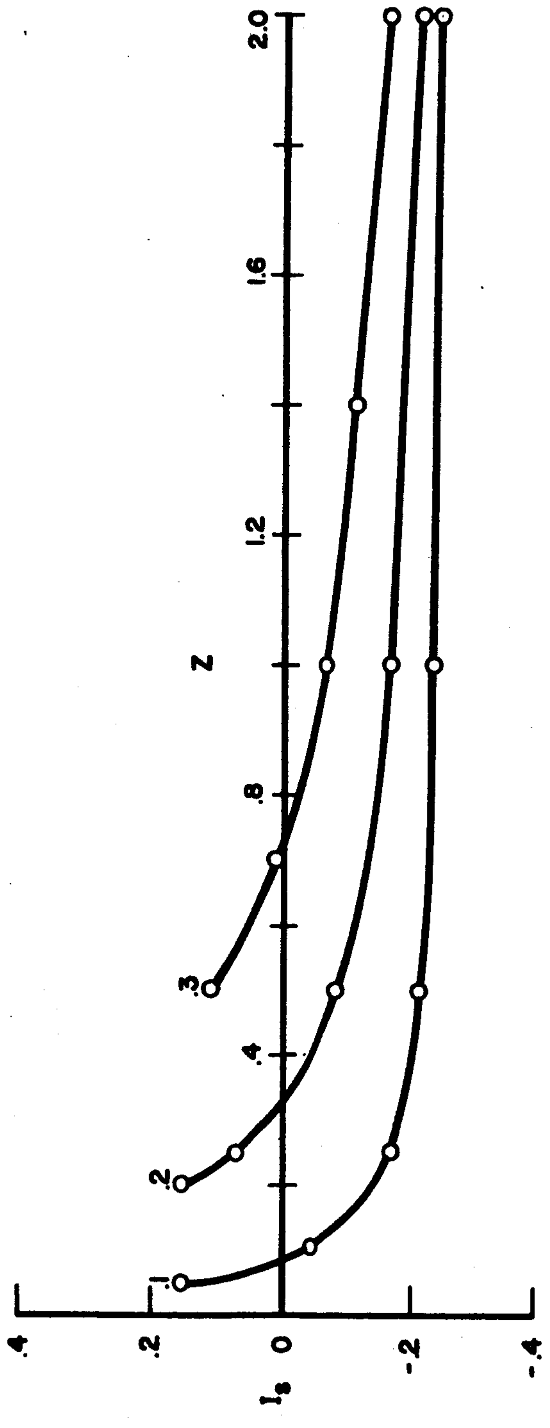

FIG. 17 – $I_s = f(Z)$, $b_1 = 0.5$, PARAMETER ζ

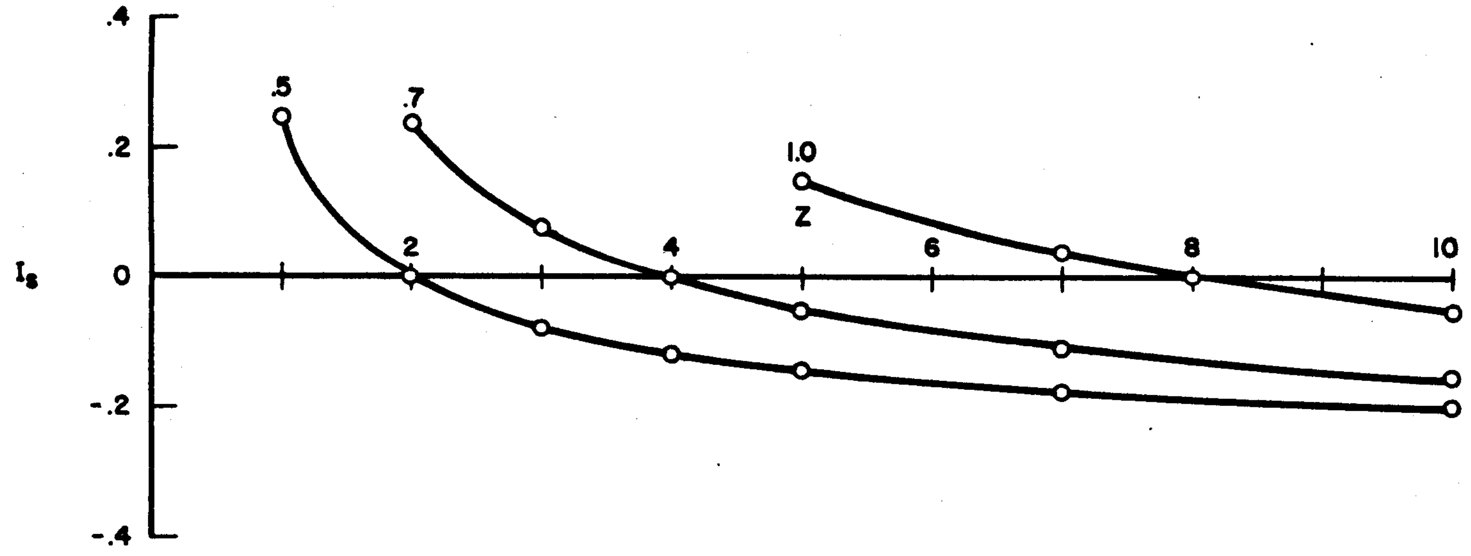

FIG. 18 — $I_s = f(Z)$, $b_1 = 0.5$, PARAMETETER ζ

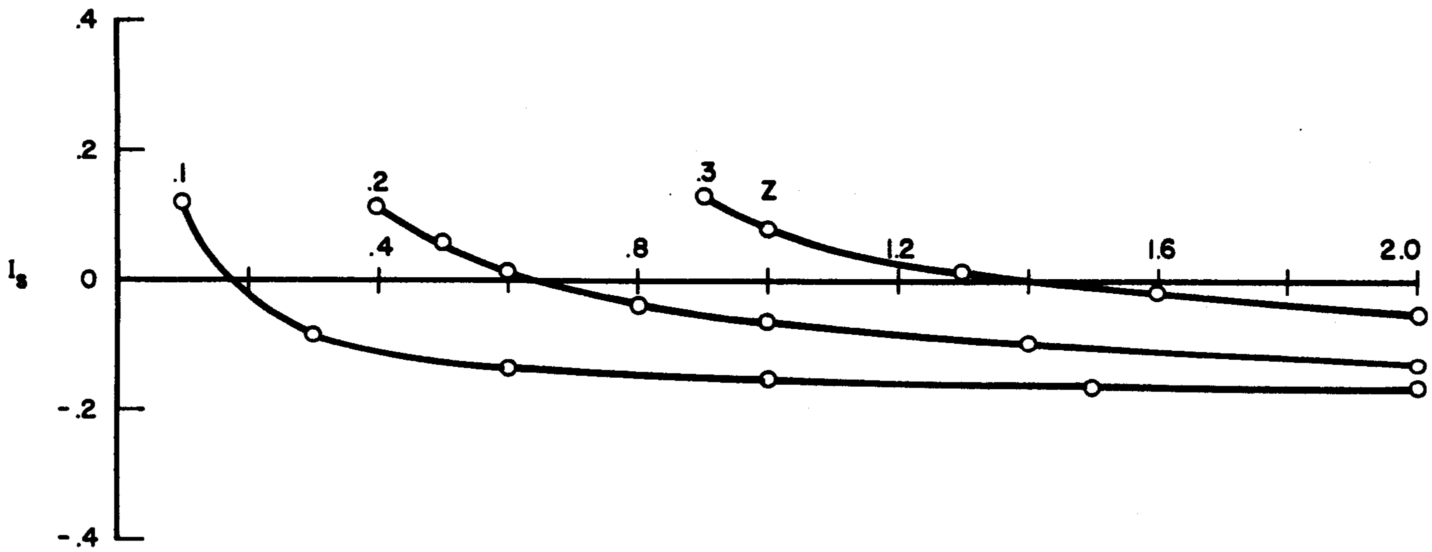

FIG. 19 – $I_s = f(Z)$, $b_1 = 0.75$, PARAMETER ζ

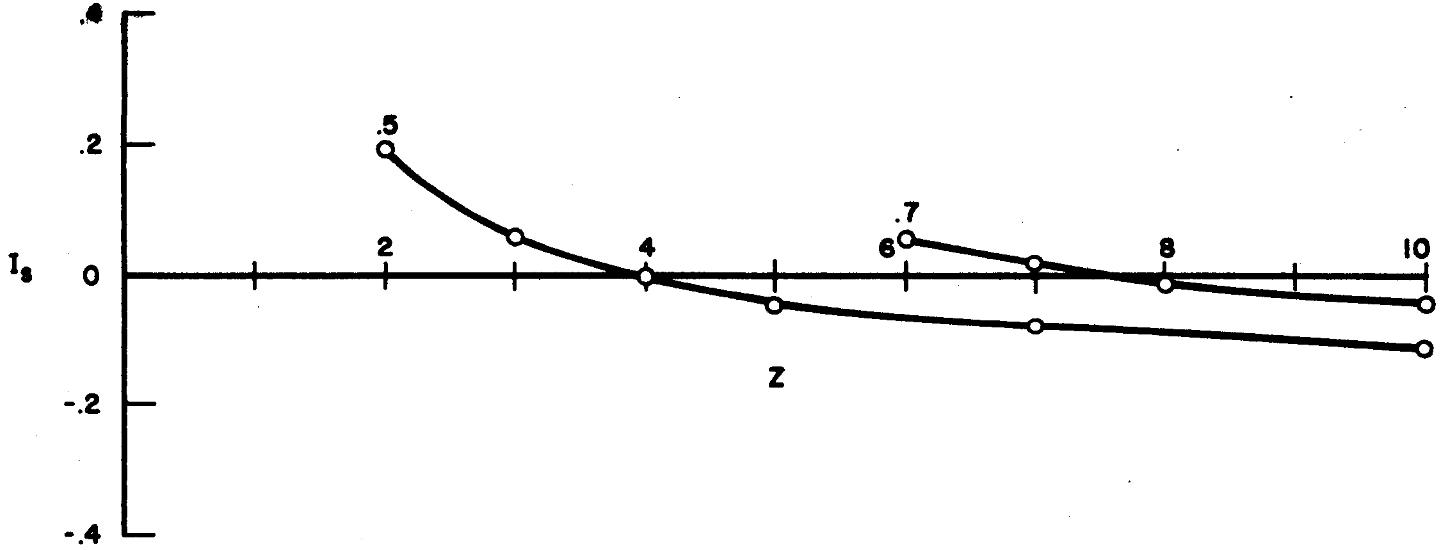

FIG. 20 — $I_s = f(Z)$, $b_1 = 0.75$, PARAMETER ζ

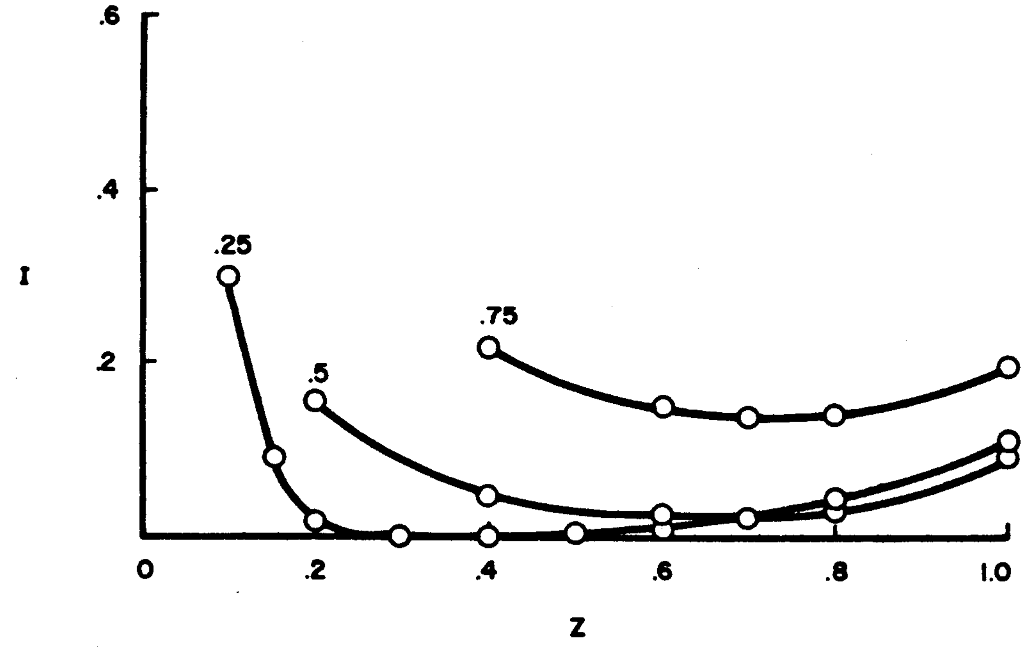

FIG. 21 – $I = f(Z)$, $\zeta = 0.2$, PARAMETER b_1

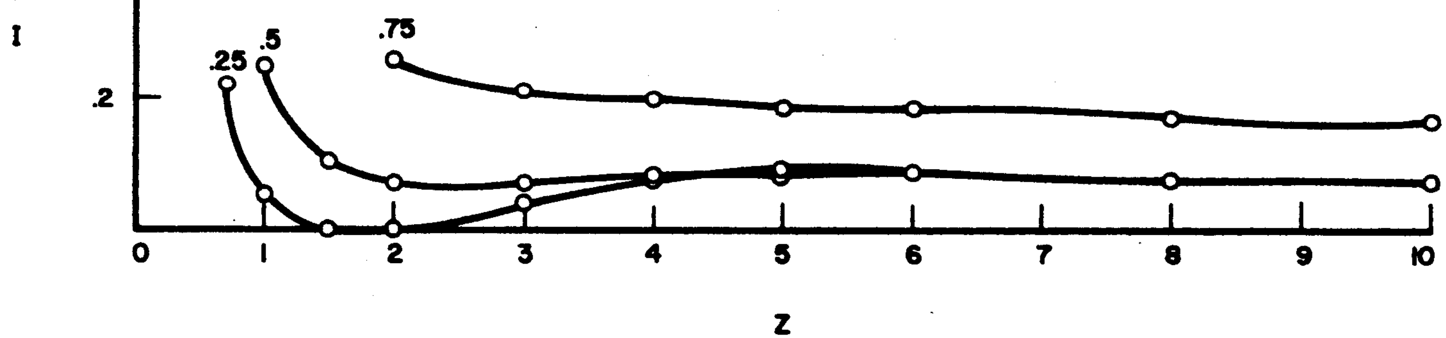

FIG. 22 — $I = f(Z), \zeta = 0.5$, PARAMETER b_1

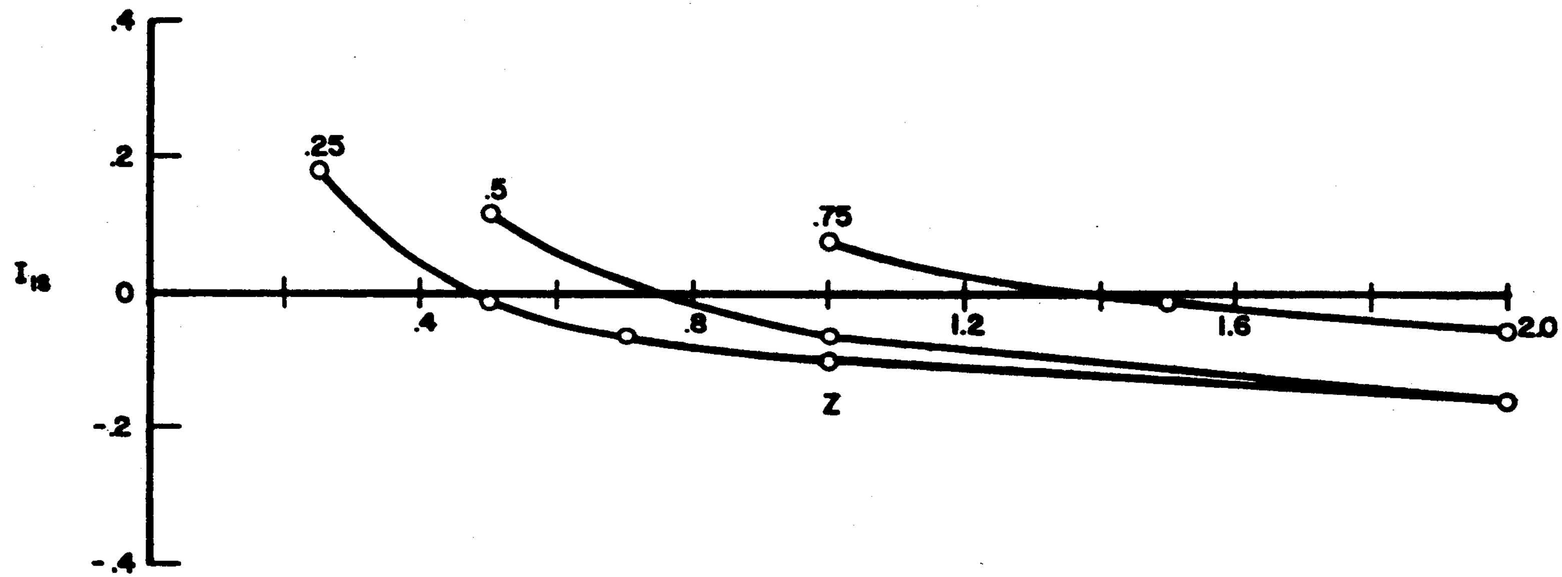

FIG. 23 — $I_{54} = f(Z)$, $\lambda = 10$, $\zeta = 0.1$, PARAMETER b_1

APPENDIX A

REFERENCES

1. Bower, John L. and Schultheiss, Peter M.: Introduction to the Design of Servomechanisms, John Wiley and Sons, Inc., New York, 1958.

2. Sokolnikoff, Ivan S.: Advanced Calculus, McGraw-Hill, Inc., New York, 1939.

3. Wegstein, J.H.: "Accelerating Convergence of Iterative Processes", Communications of the Association for Computing Machinery, Vol. 1, No. 6, page 9, June, 1958.

4. Burroughs Algebraic Compiler, Bulletin 220-21011-P, Burroughs Corp., Detroit, Michigan, 1961.

APPENDIX B

COMPUTER PROGRAMS

In the programs which follow, the variable X represents the Z of the analysis and the variable F represents the I of the analysis.

```
    BURROUGHS ALGEBRAIC COMPILER - STANDARD VERSION   5/09/61
COMMENT THIS PROGRAM EVALUATES THE FUNCTION DEFINED FOR SELECTED
VALUES OF X AND ZETA   ZETA LESS THAN 1$
ARRAY V(67,2)=(0.7,0.7,0.7,1.0,0.7,1.5,0.7,2.0,0.7,2.5,
0.7,3.5,0.7,5.0,0.7,7.0,0.7,10.0,0.5,0.47,0.5,0.5,0.5,0.75,
0.5,1.0,0.5,1.5,0.5,2.0,0.5,2.75,0.5,3.5,0.5,5.0,0.5,7.0,0.5,10.0,
0.3,0.17,0.3,0.2,0.3,0.25,0.3,0.35,0.3,0.5,0.3,
0.75,0.3,1.5,0.3,2.0,0.3,2.5,0.3,4.0,0.3,7.0,0.3,10.0,
0.2,0.08,0.2,0.1,0.2,0.12,0.2,0.15,0.2,0.2,0.2,0.25,
0.2,0.5,0.2,1.0,0.2,1.5,0.2,2.0,0.2,5.0,0.2,10.0,
0.17,0.05,0.17,0.07,0.17,0.1,0.17,0.15,0.17,0.25,0.17,0.35,
0.17,0.5,0.17,1.0,0.17,2.0,0.17,5.0,0.17,10.0,
0.1,0.03,0.1,0.05,0.1,0.1,0.1,0.2,0.1,0.3,0.1,0.4,0.1
,0.5,0.1,0.8,0.1,1.0,0.1,2.0,0.1,5.0,0.1,10.0)$
INTEGER I$ FUNCTION F(ZETA, X, A)=1.0/3-4.ZETA*2/X+
(2.ZETA*2(4.ZETA*2-1))/X*2+((1+4.ZETA*2)SINH(X)+
((3-4.ZETA*2)/A)SIN(A.X)+8.ZETA*2(EXP(-X)-COS(A.X)))/
(4.X(COSH(X)-COS(A.X)))$
WRITE ($$ FORM1 )$ FOR I=(1,1,67)$ BEGIN
A=SQRT(1-V(I,1)*2)/V(I,1)$ ZETA=V(I,1)$X=V(I,2)$
WRITE ($$ RESULTS, FORM2 ) END$
OUTPUT RESULTS (ZETA,X,F(ZETA,X,A))$
FORMAT FORM1(B25,*ZETA*,B18,*X*,B19,*F*,W),
FORM2(B20,F15.8,2(B5,F15.8),W)$FINISH$
COMPILED PROGRAM ENDS AT 0382
NEXT AVAILABLE CELL IS 1066
```

ZETA	X	F
.70000000, 00	.70000000, 00	.81003270, 00
.70000000, 00	.10000000, 01	.23053760, 00
.70000000, 00	.15000000, 01	.24202360,-01
.70000000, 00	.20000000, 01	.22313800,-02
.70000000, 00	.25000000, 01	.16600590,-01
.70000000, 00	.35000000, 01	.63514120,-01
.70000000, 00	.50000000, 01	.12608450, 00
.70000000, 00	.70000000, 01	.17825705, 00
.70000000, 00	.10000000, 02	.22074126, 00
.50000000, 00	.47000000, 00	.31793100, 00
.50000000, 00	.50000000, 00	.25003410, 00
.50000000, 00	.75000000, 00	.27954600,-01
.50000000, 00	.10000000, 01	.58142000,-03
.50000000, 00	.15000000, 01	.31021190,-01
.50000000, 00	.20000000, 01	.73621840,-01
.50000000, 00	.27500000, 01	.13807885, 00
.50000000, 00	.35000000, 01	.18931698, 00
.50000000, 00	.50000000, 01	.23386666, 00
.50000000, 00	.70000000, 01	.26187254, 00
.50000000, 00	.10000000, 02	.28333071, 00
.30000000, 00	.17000000, 00	.31228890, 00
.30000000, 00	.20000000, 00	.16000760, 00
.30000000, 00	.25000000, 00	.48418300,-01
.30000000, 00	.35000000, 00	.26910000,-03
.30000000, 00	.50000000, 00	.19886530,-01
.30000000, 00	.75000000, 00	.69282280,-01
.30000000, 00	.15000000, 01	.20090086, 00
.30000000, 00	.20000000, 01	.32233031, 00
.30000000, 00	.25000000, 01	.31834848, 00
.30000000, 00	.40000000, 01	.32290766, 00
.30000000, 00	.70000000, 01	.32807073, 00

.30000000, 00	.10000000, 02	.33018337, 00
.20000000, 00	.80000000,-01	.24999900, 00
.20000000, 00	.10000000, 00	.90006800,-01
.20000000, 00	.12000000, 00	.27787400,-01
.20000000, 00	.15000000, 00	.11218000,-02
.20000000, 00	.20000000, 00	.10035500,-01
.20000000, 00	.25000000, 00	.32487300,-01
.20000000, 00	.50000000, 00	.11735409, 00
.20000000, 00	.10000000, 01	.26302040, 00
.20000000, 00	.15000000, 01	.45495243, 00
.20000000, 00	.20000000, 01	.35204443, 00
.20000000, 00	.50000000, 01	.35686601, 00
.20000000, 00	.10000000, 02	.34566062, 00
.17000000, 00	.50000000,-01	.43035800, 00
.17000000, 00	.70000000,-01	.10610400, 00
.17000000, 00	.10000000, 00	.60890000,-02
.17000000, 00	.15000000, 00	.13169900,-01
.17000000, 00	.25000000, 00	.72425300,-01
.17000000, 00	.35000000, 00	.11286174, 00
.17000000, 00	.50000000, 00	.15163990, 00
.17000000, 00	.10000000, 01	.53041731, 00
.17000000, 00	.20000000, 01	.40090898, 00
.17000000, 00	.50000000, 01	.36328090, 00
.17000000, 00	.10000000, 02	.34915356, 00
.10000000, 00	.30000000,-01	.27789000,-01
.10000000, 00	.50000000,-01	.10006300,-01
.10000000, 00	.10000000, 00	.90035700,-01
.10000000, 00	.20000000, 00	.16065031, 00
.10000000, 00	.30000000, 00	.19217956, 00
.10000000, 00	.40000000, 00	.22517189, 00
.10000000, 00	.50000000, 00	.33919130, 00
.10000000, 00	.80000000, 00	.52700857, 00
.10000000, 00	.10000000, 01	.39576507, 00
.10000000, 00	.20000000, 01	.46172263, 00
.10000000, 00	.50000000, 01	.37703007, 00
.10000000, 00	.10000000, 02	.35514186, 00

```
         BURROUGHS ALGEBRAIC COMPILER - STANDARD VERSION   5/09/61
       COMMENT THIS PROGRAM EVALUATES THE FUNCTION DEFINED FOR
       SELECTED VALUES OF ZETA AND X   ZETA GREATER THAN 1$
       ARRAY V(9,2)=(1.5,4.0,1.5,6.0,1.5,7.0,1.5,9.0,
       1.5,10.0,1.5,100.0,2.0,8.0,2.0,10.0,2.0,100.0)$
       INTEGER I $  FUNCTION F(ZETA,X,A)=
       1.0/3-(4.ZETA*2)/X+(2.ZETA*2(4.ZETA*2-1))/X*2+
       ((1+4.ZETA*2)SINH(X)+((3-4.ZETA*2)/A)SINH(A.X)+
       8.ZETA*2(EXP(-X)-COSH(A.X)))/(4.X(COSH(X)-COSH(A.X)))$
       WRITE ($$ FORM1 )$ FOR I=(1,1,9)$ BEGIN
       A=SQRT(V(I,1)*2-1)/V(I,1)$ ZETA=V(I,1)$ X=V(I,2)$
       WRITE ($$ RESULTS, FORM2 ) END$
       OUTPUT RESULTS (ZETA,X,F(ZETA,X,A))$
       FORMAT FORM1(B25,*ZETA*,B18,*X*,B19,*F*,W),
       FORM2(B20,F15.8,2(B5,F15.8),W)$FINISH$
COMPILED PROGRAM ENDS AT 0380
NEXT AVAILABLE CELL IS 0882
```

ZETA	X	F
.15000000, 01	.40000000, 01	.39136395, 00
.15000000, 01	.60000000, 01	.64666540,-01
.15000000, 01	.70000000, 01	.23533150,-01
.15000000, 01	.90000000, 01	.54212300,-02
.15000000, 01	.10000000, 02	.92184200,-02
.15000000, 01	.10000000, 03	.27193333, 00
.20000000, 01	.80000000, 01	.25127961, 00
.20000000, 01	.10000000, 02	.92095490,-01
.20000000, 01	.10000000, 03	.22783321, 00

```
          BURROUGHS ALGEBRAIC COMPILER - STANDARD VERSION    5/09/61
        COMMENT THIS PROGRAM EVALUATES THE FUNCTION DEFINED FOR SELECTED
        VALUES OF X   ZETA EQUALS 1$
        ARRAY  V(8)=(1.5,2.0,2.5,3.0,4.0,5.0,7.0,10.0)$
        INTEGER I$ FUNCTION F(X)=1.0/3-4/X+6/X*2+(5.SINH(X)-X+8(EXP(-X)-1))/
        4.X(COSH(X)-1)$
        WRITE ($$FORM1)$ FOR I=(1,1,8)$ BEGIN X=V(I)$ WRITE
        ($$RESULTS,FORM2) ENDS OUTPUT RESULTS (X, F(X))$
        FORMAT FORM1(B26,*X*,B20,*F*,W0),
        FORM2(B20,F15.8,B5,F15.8,W0)$FINISH$
COMPILED PROGRAM ENDS AT 0296
NEXT AVAILABLE CELL IS 0735

             X                         F
         .15000000, 01            .69459442, 00
         .20000000, 01            .25043737, 00
         .25000000, 01            .90966800,-01
         .30000000, 01            .29564910,-01
         .40000000, 01            .43340900,-02
         .50000000, 01            .17883420,-01
         .70000000, 01            .62272800,-01
         .10000000, 02            .11830381, 00
```

```
     BURROUGHS ALGEBRAIC COMPILER - STANDARD VERSION   2/20/61
COMMENT THIS PROGRAM EVALUATES THE FUNCTION DEFINED FOR SELECTED
VALUES OF X AND ZETA   ZETA LESS THAN 1$
ARRAY V(81,2)=(0.1,0.02,0.1,0.6,0.1,0.7,0.1,0.9,0.1,1.2,
0.1,0.64,0.1,1.1,0.1,1.3,0.1,1.5,0.1,1.6,0.1,1.8,
0.1,1.9,0.1,2.5,0.1,3.0,0.1,3.5,0.1,4.0,0.1,6.0,0.1,8.0,
0.1,1.4,0.1,1.7,0.1,3.0,0.1,4.0,0.1,6.0,0.1,8.0,0.1,9.0,0.1,100.0,
0.17,0.6,0.17,0.8,0.17,1.1,0.17,1.2,0.17,1.4,0.17,1.6,0.17,1.8,
0.17,3.0,0.17,4.0,0.17,6.0,0.17,8.0,0.17,9.0,0.17,100.0,
0.17,1.05,0.17,1.3-0.17,1.5,0.17,1.7,0.17,1.9,
0.17,2.5,0.17,3.0,u.17,3.5,0.17,4.0,0.17,6.0,0.17,7.0,0.17,8.0,
0.2,0.7,0.2,1.2,0.2,1.4,0.2,1.6,0.2,1.8,0.2,100.0,0.2,1.7,
0.2,3.0,0.2,3.5,0.2,4.0,0.2,4.5,0.2,6.0,0.2,7.0,0.2,8.0,0.2,11.0,
0.2,1.3,0.2,1.5,0.2,1.7,0.2,1.9,0.2,2.5,
0.3,1.0,0.3,1.2,0.3,1.4,0.3,1.7,0.3,1.9,0.3,100.0,
0.5,100.0,0.7,0.95,0.7,1.2,0.7,100.0)$
INTEGER I$ FUNCTION F(ZETA, X, A)=1.0/3-4.ZETA*2/X+
(2.ZETA*2(4.ZETA*2-1))/X*2+((1+4.ZETA*2)SINH(X)+
((3-4.ZETA*2)/A)SIN(A.X)+8.ZETA*2(EXP(-X)-COS(A.X)))/
(4.X(COSH(X)-COS(A.X)))$
WRITE ($$ FORM1 )$ FOR I=(1,1,81)$ BEGIN
A=SQRT(1-V(I,1)*2)/V(I,1)$ ZETA=V(I,1)$X=V(I,2)$
WRITE ($$ RESULTS, FORM2 ) END$
OUTPUT RESULTS (ZETA,X,F(ZETA,X,A))$
FORMAT FORM1(B25,*ZETA*,B18,*X*,B19,*F*,W),
FORM2(B20,F15.8,2(B5,F15.8),W)$FINISH$
COMPILED PROGRAM ENDS AT 0382
NEXT AVAILABLE CELL IS 1094
```

ZETA	X	F
.10000000, 00	.20000000,-01	.24988000, 00
.10000000, 00	.60000000, 00	.11712042, 01
.10000000, 00	.70000000, 00	.94877698, 00
.10000000, 00	.90000000, 00	.42125441, 00
.10000000, 00	.12000000, 01	.56873147, 00
.10000000, 00	.64000000, 00	.14922824, 01
.10000000, 00	.11000000, 01	.42951175, 00
.10000000, 00	.13000000, 01	.62849819, 00
.10000000, 00	.15000000, 01	.43418897, 00
.10000000, 00	.16000000, 01	.41081606, 00
.10000000, 00	.18000000, 01	.45886906, 00
.10000000, 00	.19000000, 01	.48902077, 00
.10000000, 00	.25000000, 01	.43323578, 00
.10000000, 00	.30000000, 01	.40170972, 00
.10000000, 00	.35000000, 01	.39041390, 00
.10000000, 00	.40000000, 01	.38658095, 00
.10000000, 00	.60000000, 01	.36926832, 00
.10000000, 00	.80000000, 01	.36051803, 00
.10000000, 00	.14000000, 01	.50381724, 00
.10000000, 00	.17000000, 01	.41938053, 00
.10000000, 00	.30000000, 01	.40170972, 00
.10000000, 00	.40000000, 01	.38658095, 00
.10000000, 00	.60000000, 01	.36926832, 00
.10000000, 00	.80000000, 01	.36051803, 00
.10000000, 00	.90000000, 01	.35754269, 00
.10000000, 00	.10000000, 03	.33553140, 00
.17000000, 00	.60000000, 00	.17303627, 00
.17000000, 00	.80000000, 00	.24521001, 00
.17000000, 00	.11000000, 01	.65317689, 00

.17000000, 00	.12000000, 01	.58285529, 00
.17000000, 00	.14000000, 01	.42641530, 00
.17000000, 00	.16000000, 01	.37254896, 00
.17000000, 00	.18000000, 01	.36866082, 00
.17000000, 00	.30000000, 01	.37840254, 00
.17000000, 00	.40000000, 01	.36914959, 00
.17000000, 00	.60000000, 01	.35892969, 00
.17000000, 00	.80000000, 01	.35294065, 00
.17000000, 00	.90000000, 01	.35084792, 00
.17000000, 00	.10000000, 03	.33496121, 00
.17000000, 00	.10500000, 01	.61852996, 00
.17000000, 00	.13000000, 01	.48765674, 00
.17000000, 00	.15000000, 01	.39124325, 00
.17000000, 00	.17000000, 01	.36555494, 00
.17000000, 00	.19000000, 01	.38136788, 00
.17000000, 00	.25000000, 01	.39181558, 00
.17000000, 00	.30000000, 01	.37840254, 00
.17000000, 00	.35000000, 01	.37834946, 00
.17000000, 00	.40000000, 01	.36914959, 00
.17000000, 00	.60000000, 01	.35892969, 00
.17000000, 00	.70000000, 01	.35557153, 00
.17000000, 00	.80000000, 01	.35294065, 00
.20000000, 00	.70000000, 00	.15815771, 00
.20000000, 00	.12000000, 01	.46304853, 00
.20000000, 00	.14000000, 01	.50257436, 00
.20000000, 00	.16000000, 01	.41414260, 00
.20000000, 00	.18000000, 01	.36719355, 00
.20000000, 00	.10000000, 03	.33462660, 00
.20000000, 00	.17000000, 01	.38559517, 00
.20000000, 00	.30000000, 01	.36922659, 00
.20000000, 00	.35000000, 01	.36195095, 00
.20000000, 00	.40000000, 01	.36397330, 00
.20000000, 00	.45000000, 01	.35784176, 00
.20000000, 00	.60000000, 01	.35294988, 00
.20000000, 00	.70000000, 01	.35049043, 00
.20000000, 00	.80000000, 01	.34854681, 00
.20000000, 00	.11000000, 02	.34459537, 00
.20000000, 00	.13000000, 01	.52331291, 00
.20000000, 00	.15000000, 01	.45495243, 00
.20000000, 00	.17000000, 01	.38559517, 00
.20000000, 00	.19000000, 01	.35651855, 00
.20000000, 00	.25000000, 01	.38513420, 00
.30000000, 00	.10000000, 01	.10897499, 00
.30000000, 00	.12000000, 01	.13932560, 00
.30000000, 00	.14000000, 01	.17686938, 00
.30000000, 00	.17000000, 01	.25843069, 00
.30000000, 00	.19000000, 01	.30848470, 00
.30000000, 00	.10000000, 03	.33312180, 00
.50000000, 00	.10000000, 03	.32833333, 00
.70000000, 00	.95000000, 00	.28268828, 00
.70000000, 00	.12000000, 01	.10056256, 00
.70000000, 00	.10000000, 03	.32122740, 00

```
     BURROUGHS ALGEBRAIC COMPILER - STANDARD VERSION    5/09/61
   COMMENT THIS PROGRAM EVALUATES THE FUNCTION DEFINED FOR
   SELECTED VALUES OF ZETA AND X   ZETA GREATER THAN 1$
   ARRAY V(9,2)=(1.5,3.0,1.5,5.0,1.5,8.0,1.5,12.0,1.5,16.0,
   2.0,5.5,2.0,7.0,2.0,12.0,2.0,16.0)$
   INTEGER I $  FUNCTION F(ZETA,X,A)=
   1.0/3-(4.ZETA*2)/X+(2.ZETA*2(4.ZETA*2-1))/X*2+
   ((1+4.ZETA*2)SINH(X)+((3-4.ZETA*2)/A)SINH(A.X)+
   8.ZETA*2(EXP(-X)-COSH(A.X)))/(4.X(COSH(X)-COSH(A.X)))$
   WRITE ($$ FORM1 )$ FOR I=(1,1,9)$ BEGIN
   A=SQRT(V(I,1)*2-1)/V(I,1)$ ZETA=V(I,1)$ X=V(I,2)$
   WRITE ($$ RESULTS, FORM2 ) END$
   OUTPUT RESULTS (ZETA,X,F(ZETA,X,A))$
   FORMAT FORM1(B25,*ZETA*,B18,*X*,B19,*F*,W),
   FORM2(B20,F15.8,2(B5,F15.8),W)$FINISH$
COMPILED PROGRAM ENDS AT 0380
NEXT AVAILABLE CELL IS 0882
```

ZETA	X	F
.15000000, 01	.30000000, 01	.10003107, 01
.15000000, 01	.50000000, 01	.16136380, 00
.15000000, 01	.80000000, 01	.81231000,-02
.15000000, 01	.12000000, 02	.25143590,-01
.15000000, 01	.16000000, 02	.63370320,-01
.20000000, 01	.55000000, 01	.91167710, 00
.20000000, 01	.70000000, 01	.41420418, 00
.20000000, 01	.12000000, 02	.30851370,-01
.20000000, 01	.16000000, 02	.54345900,-02

```
     BURROUGHS ALGEBRAIC COMPILER - STANDARD VERSION   2/20/61
COMMENT THIS PROGRAM LOCATES A ROOT FOR THE FUNCTION
DEFINED BY USING WEGSTEIN S IMPROVED ITERATION PROCESS
ZETA LESS THAN 1
ROY WAAS$
ARRAY V(4,2)=(0.7,1.5,0.5,1.0,0.3,0.35,0.2,0.15)$
INTEGER I$ FUNCTION F(X,ZETA,A)=4.ZETA*2/X*2
-(4.ZETA*2)(4.ZETA*2-1)/(X*3)+(((1+4.ZETA*2)
COSH(X)+(3-4.ZETA*2)(COS(A.X))+8.ZETA*2.(A.SIN(A.X)-EXP(-X)))
X(COSH(X)-COS(A.X))-((1+4.ZETA*2)SINH(X)
+((3-4.ZETA*2)/A)SIN(A.X)+(8.ZETA*2)(EXP(-X)-COS(A.X)))(
COSH(X)+X.SINH(X)-COS(A.X)+A.X.SIN(A.X)))/4
((X(COSH(X)-COS(A.X)))*2)$
WRITE ($$ FORM1 )$ FOR I= (1,1,4)$ BEGIN
A=SQRT(1-V(I,1)*2)/V(I,1)$ ZETA=V(I,1)$ X=V(I,2)$
ALPHA=X$BETA=(F(X,ZETA,A))+X$GAMMA=BETA$GO SECOND$
FIRST..ALPHA=BETA$ BETA=TAU$ GAMMA=LAMBDA$
SECOND..X=BETA$ LAMBDA=(F(X,ZETA,A))+X$
TAU=(LAMBDA.ALPHA-GAMMA.BETA)/(LAMBDA+ALPHA-GAMMA-BETA)$
IF ABS(TAU-BETA) GTR 7.0**-5$ GO TO FIRST$X=TAU$
WRITE($$RESULTS,FORM2) END$
OUTPUT RESULTS(ZETA,X,F(X,ZETA,A),1/X)$
FORMAT FORM1(B25,*ZETA*,B13,*X*,B14,*F*,B14,*1/X*,W),
FORM2(B20,4(F15.8),W)$FINISH$
COMPILED PROGRAM ENDS AT 0531
NEXT AVAILABLE CELL IS 1133
```

ZETA	X	F	1/X
.70000000, 00	.19324758, 01	.28300000,-05	.51747090, 00
.50000000, 00	.99530334, 00	.90000000,-05	.10047188, 01
.30000000, 00	.35978498, 00	-.37000000,-05	.27794378, 01
.20000000, 00	.15998260, 00	-.80000000,-05	.62506797, 01

```
    BURROUGHS ALGEBRAIC COMPILER - STANDARD VERSION   2/20/61
COMMENT THIS PROGRAM LOCATES A ROOT FOR THE FUNCTION
DEFINED BY USING WEGSTEIN S IMPROVED ITERATION PROCESS
ZETA LESS THAN 1
ROY WAAS$
ARRAY V(2,2)=(0.17,0.12,0.1,0.04)$
INTEGER I$ FUNCTION F(X,ZETA,A)=4.ZETA*2/X*2
-(4.ZETA*2)(4.ZETA*2-1)/(X*3)+(((1+4.ZETA*2)
COSH(X)+(3-4.ZETA*2)(COS(A.X))+8.ZETA*2.(A.SIN(A.X)-EXP(-X)))
X(COSH(X)-COS(A.X))-((1+4.ZETA*2)SINH(X)
+((3-4.ZETA*2)/A)SIN(A.X)+(8.ZETA*2)(EXP(-X)-COS(A.X)))(
COSH(X)+X.SINH(X)-COS(A.X)+A.X.SIN(A.X)))/4
((X(COSH(X)-COS(A.X)))*2)$
WRITE ($$ FORM1 )$ FOR I= (1,1,2)$ BEGIN
A=SQRT(1-V(I,1)*2)/V(I,1)$ ZETA=V(I,1)$ X=V(I,2)$
ALPHA=X$BETA=(F(X,ZETA,A))1.0**-3+X$GAMMA=BETA$GO SECOND$
FIRST..ALPHA=BETA$ BETA=TAU$ GAMMA=LAMBDA$
SECOND..X=BETA$ LAMBDA=(F(X,ZETA,A))1.0**-3+X$
TAU=(LAMBDA.ALPHA-GAMMA.BETA)/(LAMBDA+ALPHA-GAMMA-BETA)$
IF ABS(TAU-BETA) GTR 7.0**-5$ GO TO FIRST$X=TAU$
WRITE($$RESULTS,FORM2) END$
OUTPUT RESULTS(ZETA,X,F(X,ZETA,A),1/X)$
FORMAT FORM1(B25,*ZETA*,B13,*X*,B14,*F*,B14,*1/X*,W),
FORM2(B20,4(F15.8),W)$FINISH$
COMPILED PROGRAM ENDS AT 0533
NEXT AVAILABLE CELL IS 1131

         ZETA               X                 F                1/X
     .17000000, 00    .11560599, 00    .47800000,-03    .86500708, 01
     .10000000, 00    .40000000,-01    .84000000,-03    .25000000, 02
```

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    BURROUGHS ALGEBRAIC COMPILER - STANDARD VERSION   2/20/61
COMMENT THIS PROGRAM LOCATES A ROOT FOR THE FUNCTION
DEFINED BY USING WEGSTEIN S IMPROVED ITERATION PROCESS
ZETA GREATER THAN 1
ROY WAAS$
ARRAY V(2,2)=(1.5,9.0,2.0,16.0)$
INTEGER I$ FUNCTION F(X,ZETA,A)=(4.ZETA*2)/(X*2)
-(4.ZETA*2)(4.ZETA*2-1)/(X*3)+((X(COSH(X)-
COSH(A.X)))((1+4.ZETA*2)COSH(X)+(3-4.ZETA*2)COSH(A.X)
-8.ZETA*2(EXP(-X)+A.SINH(A.X)))-((1+4.ZETA*2)
SINH(X)+((3-4.ZETA*2)/A)SINH(A.X)+8.ZETA*2
(EXP(-X)-COSH(A.X)))(X(SINH(X)-A.SINH(A.X))+
COSH(X)-COSH(A.X)))/4((X(COSH(X)-COSH(A.X)))*2)$
WRITE($$FORM1)$ FOR I=(1,1,2)$ BEGIN
A=SQRT(V(I,1)*2-1)/V(I,1)$ ZETA=V(I,1)$ X=V(I,2)$
ALPHA=X$BETA=(F(X,ZETA,A))1.0**3+X $GAMMA=BETA$GO SECOND$
FIRST..ALPHA=BETA$ BETA=TAU$ GAMMA=LAMBDA$
SECOND..X=BETA$ LAMBDA=(F(X,ZETA,A))1.0**3+X$
TAU=(LAMBDA.ALPHA-GAMMA.BETA)/(LAMBDA+ALPHA-GAMMA-BETA)$
IF ABS(TAU-BETA) GTR 2.0**-4$ GO TO FIRST$X=TAU$
WRITE($$RESULTS,FORM2) END$
OUTPUT RESULTS(ZETA,X,F(X,ZETA,A),1/X)$
FORMAT FORM1(B25,*ZETA*,B13,*X*,B14,*F*,B14,*1/X*,W),
FORM2(B20,4(F15.8),W)$FINISH$
COMPILED PROGRAM ENDS AT 0530
NEXT AVAILABLE CELL IS 1062

      ZETA             X               F                1/X
 .15000000, 01   .88118988, 01  -.88200000,-06   .11348291, 00
 .20000000, 01   .15691097, 02   .19920000,-06   .63730407,-01
```

```
    BURROUGHS ALGEBRAIC COMPILER - STANDARD VERSION   2/20/61
COMMENT THIS PROGRAM LOCATES A ROOT FOR THE FUNCTION
DEFINED BY USING WEGSTEIN S IMPROVED ITERATION PROCESS
ZETA EQUALS 1
ROY WAAS$
FUNCTION F(X)=4/X*2-12/X*3+(X(COSH(X)-1)(5COSH(X)-1-8EXP(-X))
-(5SINH(X)-X+8(EXP(-X)-1))(X.SINH(X)+COSH(X)-1))
/4X*2(COSH(X)-1)*2$X=3.0$
ALPHA=X$BETA=F(X)+X$GAMMA=BETA$GO TO SECOND$
FIRST..ALPHA=BETA$BETA=TAU$
GAMMA=LAMBDA$SECOND..X=BETA$LAMBDA=(F(X))+X$
TAU=(LAMBDA.ALPHA-GAMMA.BETA)/(LAMBDA+ALPHA-GAMMA-BETA)$
IF ABS(TAU-BETA) GTR 1.0**-4$GO FIRST$X=TAU$WRITE($$RESULTS,FORM)$
OUTPUT RESULTS(X,F(X),1/X)$
FORMAT FORM(B20,*X=*,F14.8,B2,*F(X)=*,F14.8,B2,*1/X=*,F14.8,W)$
FINISH$
COMPILED PROGRAM ENDS AT 0373
NEXT AVAILABLE CELL IS 0849

X= .39099365, 01  F(X)= .15770000,-05  1/X= .25575862, 00
```

```
  000 19006       ALGOL                                                                          MAY    02    15 55
0200                      BAC-220 STANDARD VERSION   2/1/62
0200        COMMENT THIS PROGRAM EVALUATES THE FUNCTIONS DEFINED FOR
0200        SELECTED VALUES OF X.E.AND L$
0200        SUBROUTINE EVAL$BEGIN
0205        A=SQRT(1-L*2)/L$
0212        P=2-E$G=(1-E)*2+A*2
0224        $H=E-1●A*2$R=COSH(X)-COS(A.X)$T1=1/3.0$
0243        T2=1/TANH(E.X/2).2E.X.L*4.G*2$
0267        T3=(A.L*2.E*2(H*2-A*2.P*2-2P.H)SIN(A.X)-L*2.E*2(H*2-A*2.P*2+2A*2
0322        .P.H)SINH(X))/4R.A*2.X.G*2$
0344        T4=E*2(H*2+A*2.P*2)SINH(X)/4X.R.A*2.G*2$
0378        T5=-2/E.L*2.X.G$T6=2.E.L*2(P-H)/X.G
0404        $T7=(2/E.X.L*2.G)(1/(1-EXP(E.X))+1/E.X)$T8=2E.L*4(2H-P(1-A*2))/X*2.G$
0458        T9=(A.E.L*2(H-P)(EXP(-X)-COS(A.X))-L*2.E(H+A*2.P)SIN(A.X))/A.X.G.R$
0504        T10=(E(3A*2+E*2-1)SIN(A.X)-(A*2+E*2-3)A.E(COS(A.X)(1/TANH(E.X/2))+
0546        (EXP(-X)+EXP(X(1+E)))/(1-EXP(E.X))))/A.X.R.L*2.G*2(G+4E)$
0590        F=T1+T2+T3+T4+T5+T6+T7+T8+T9+T10$RETURN END EVAL$
0603        SUBROUTINE EVAL2$BEGIN
0608        A=SQRT(1-L*2)/L$
0615        P=2-E$G=(1-E)*2+A*2
0627        $H=E-1+A*2$R=COSH(X)-COS(A.X)$T1=1/3.0$
0646        T2=1/2E.X.L*4.G*2$
0662        T3=(A.L*2.E*2(H*2-A*2.P*2-2P.H)SIN(A.X)-L*2.E*2(H*2-A*2.P*2+2A*2
0717        .P.H)SINH(X))/4R.A*2.X.G*2$
0739        T4=E*2(H*2+A*2.P*2)SINH(X)/4X.R.A*2.G*2$
0773        T5=-2/E.L*2.X.G$T6=2.E.L*2(P-H)/X.G$
0799        T7=2/E*2.X*2.L*2.G$T8=2E.L*4(2H-P(1-A*2))/X*2.G$
0840        T9=(A.E.L*2(H-P)(EXP(-X)-COS(A.X))-L*2.E(H+A*2.P)SIN(A.X))/A.X.G.R$
0886        T10=(E(3A*2+E*2-1)SIN(A.X)-A.E(A*2+E*2-3)(COS(A.X)-EXP(X)))/
0937        A.L*2.X.R.G*2(G+4E)$
0943        F=T1+T2+T3+T4+T5+T6+T7+T8+T9+T10$RETURN END EVAL2$
0956        SUBROUTINE CRIT$BEGIN P=2-E$H=E-1$R=COSH(X)-1$
0969        V1=1/3.0$V2=1/TANH(E.X/2).2E.X.H*4$
0991        V3=(E*2.H(H-2P)X+2E*2.P(P-H)SINH(X))/4X.R.H*4$V4=-2/E.X.H*2$
1037        V5=2E(P-H)/X.H*2$V6=(2/E.X.H*2)(1/(1-EXP(E.X))+1/E.X)$
1078        V7=2E(2H-P)/X*2.H*2$V8=(E(H-P)(EXP(-X)-1)-E.H.X)/X.R.H*2$
1119        V9=(E(E*2-1)X-(E*2-3)E(1/TANH(E.X/2)+(EXP(-X)+EXP(X(1+E)))/
1154        (1-EXP(E.X))))/X.R.H*4(H*2+4.E)$
1184        I=V1+V2+V3+V4+V5+V6+V7+V8+V9$
1194        RETURN END CRIT$
1196        SUBROUTINE CRIT2$BEGIN P=2-E$H=E-1$R=COSH(X)-1$
1209        V1=1/3.0$V2=1/2E.X.H*4$
1223        V3=(E*2.H(H-2P)X+2E*2.P(P-H)SINH(X))/4X.R.H*4$V0=-2/E.X.H*2$
1269        V5=2E(P-H)/X.H*2$V6=2/E*2.X*2.H*2$
1297        V7=2E(2H-P)/X*2.H*2$V8=(E(H-P)(EXP(-X)-1)-E.H.X)/X.R.H*2$
1338        V9=(E(E*2-1)X-(E*2-3)E(1-EXP(X)))/X.R.H*4(H*2+4E)$
1375        I=V1+V2●V3+V4+V5+V6+V7+V8+V9$
1385        RETURN END CRIT2$
1387        SUBROUTINE OVER$BEGIN
1392        A=SQRT(L*2-1)/L$
1397        P=2-F$G=(1-E)*2-A*2
1411        $H=E-1-A*2$R=COSH(X)-COSH(A.X)$T1=1/3.0$
1432        T2=1/TANH(E.X/2).2E.X.L*4.G*2$
1456        T3=(A.L*2.E*2(H*2+A*2.P*2-2P.H)SINH(A.X)+L*2.E*2(H*2+A*2.P*2-2A*2
1511        .P.H)SINH(X))/4R.A*2.X.G*2$
1533        T4=E*2(A*2.P*2-H*2)SINH(X)/4X.R.A*2.G*2$
1567        T5=-2/E.L*2.X.G$T6=2.E.L*2(P-H)/X.G
1593        $T7=(2/E.X.L*2.G)(1/(1-EXP(E.X))+1/E.X)$T8=2E.L*4(2H-P(1+A*2))/X*2.G$
1647        T9=(A.E.L*2(H-P)(EXP(-X)-COSH(A.X))-L*2.E(H-A*2.P)SINH(A.X))/A.X.G.R$
```

```
1693      T10=(E(E*2-3.A*2-1)SINH(A.X)-(E*2-A*2-3)A.E(COSH(A.X)(1/TANH(E.X/2))*
1735      (EXP(-X)+EXP(X(1+E)))/(1-EXP(E.X))))/A.X.R.L*2.G*2(G+4E)$
1779      F=T1+T2+T3+T4+T5+T6+T7+T8+T9+T10$RETURN END OVER$
1792      SUBROUTINE OVER2$BEGIN
1797      A=SQRT(L*2-1)/L$
1802      P=2-E$G=(1-E)*2-A*2
1816      $H=E-1-A*2$R=COSH(X)-COSH(A.X)$T1=1/3.0$
1837      T2=1/2E.X.L*4.G*2$
1853      T3=(A.L*2.E*2(H*2+A*2.P*2-2P.H)SINH(A.X)+L*2.E*2(H*2+A*2.P*2-2H*2
1908      .P.H)SINH(X))/4R.A*2.X.G*2$
1930      T4=E*2(A*2.P*2-H*2)SINH(X)/4X.R.A*2.G*2$
1964      T5=-2/E.L*2.X.G$T6=2.E.L*2(P-H)/X.G$
1990      T7=2/E*2.X*2.L*2.G$T8=2E.L*4(2H-P(1+A*2))/X*2.G$
2031      T9=(A.E.L*2(H-P)(EXP(-X)-COSH(A.X))-L*2.E(H-A*2.P)SINH(A.X))/A.X.G.R$
2077      T10=(E(E*2-3.A*2-1)SINH(A.X)-A.E(E*2-A*2-3)(COSH(A.X)-EXP(X)))/
2128      A.L*2.X.R.G*2(G+4E)$
2134      F=T1+T2+T3+T4+T5+T6+T7+T8+T9+T10$RETURN END OVER2$
2147      SUBROUTINE TEST$
2157      BEGIN EITHER IF (L LSS 1.0) AND (E.X GTR 20.0)$ENTER EVAL2$
2171      OR IF (L LSS 1.0) AND (E.X LEQ 20.0)$ENTER EVAL$
2194      OR IF (L EQL 1.0)
2202      AND (E.X LEQ 20.0)
2212      $ENTER CRIT$OR IF (L EQL 1.0) AND (E.X GTR 20.0)$
2238      ENTER CRIT2$OR IF (L GTR 1.0) AND (E.X GTR 20.0)$ENTER OVER2$
2261      OTHERWISE$ENTER OVER$EITHER IF L EQL 1.0$WRITE ($$SECO,FORM3)$
2277      OTHERWISE$WRITE ($$RESULTS,FORM2)$RETURN END TEST$
2288      WRITE ($$FORM1)$
2292      FOR E=30.0$BEGIN FOR L=2.0$BEGIN FOR X=(15.5,0.1,16.5)$
2313      BEGIN ENTER TEST$END END END$
2319      FOR E=10.0$BEGIN FOR L=2.0$BEGIN FOR X=(15.6,0.1,17.1)$
2340      BEGIN ENTER TEST$END END END$
2346      FOR E=5.0$ BEGIN FOR L=1.1$BEGIN FOR X=5.1,5.2$
2362      BEGIN ENTER TEST$END END END$
2370      FOR E=5.0$BEGIN FOR L=2.0$BEGIN FOR X=(16.1,0.1,18.0)$
2391      BEGIN ENTER TEST$END END END$
2397      OUTPUT RESULTS(E,L,X,A,P,G,H,R,T1,T2,T3,T4,T5,T6,T7,T8,T9,T10,F),
2458      SECO(E,L,X,P,H,R,V1,V2,V3,V4,V5,V6,V7,V8,V9,I)$
2510      FORMAT FORM1 (B3,*E*,B4,*L*,B3,*X*,B4,*A*,B4,*P*,B4,*G*,B4,*H*,
2510      B5,*R*,B6,*T1*,B6,*T2*,B4,*T3*,B5,*T4*,B6,*T5*,B4,*T6*,B6,
2510      *T7*,B4,*T8*,B5,*T9*,B5,*T10*,B5,*F*,W),
2510      FORM2 (2X5.2,5X5.2,12S7.5,W4),
2510      FORM3 (2X5.2,X5.2,2X10.2,4S7.5,B7,7S7.5,W4)
2564      $FINISH$
COMPILED PROGRAM ENDS AT 2565
PROGRAM VARIABLES BEGIN AT 4319

   E     L   X     A    P    G    H      R       T1      T2     T3      T4        T5    T6       T7      T8      T9      T10     F

30.00 2.0015.50  .86    *    *28.25      * .33333 .00000 .36292-.00658-.00000-1.0365 .00000 .50170-.14937 .00000 .00544

30.00 2.0015.60  .86    *    *28.25      * .33333 .00000 .35937-.00652-.00000-1.0299 .00000 .49529-.14616 .00000 .00540

30.00 2.0015.70  .86    *    *28.25      * .33333 .00000 .35589-.00647-.00000-1.0233 .00000 .48900-.14303 .00000 .00537

30.00 2.0015.80  .86    *    *28.25      * .33333 .00000 .35248-.00642-.00000-1.0168 .00000 .48283-.13997 .00000 .00537

30.00 2.0015.90  .86    *    *28.25      * .33333 .00000 .34912-.00637-.00000-1.0104 .00000 .47678-.13699 .00000 .00539

30.00 2.0016.00  .86    *    *28.25      * .33333 .00000 .34583-.00631-.00000-1.0041 .00000 .47084-.13408 .00000 .00543
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```
30.00 2.0016.10  .86    *     *28.25       * .33333 .00000 .34260-.00626-.00000-.99792 .00000 .46501-.13125 .00000 .00549
30.00 2.0016.20  .86    *     *28.25       * .33333 .00000 .33942-.00621-.00000-.99176 .00000 .45928-.12848 .00000 .00558
30.00 2.0016.30  .86    *     *28.25       * .33333 .00000 .33630-.00617-.00000-.98568 .00000 .45366-.12577 .00000 .00568
30.00 2.0016.40  .86    *     *28.25       * .33333 .00000 .33324-.00612-.00000-.97967 .00000 .44815-.12313 .00000 .00579
30.00 2.0016.50  .86    *     *28.25       * .33333 .00000 .33023-.00607-.00000-.97373 .00000 .44273-.12056 .00000 .00593
10.00 2.0015.60  .86-8.0080.25 8.25        * .33333 .00000 .36595-.00759-.00003-1.0384 .00000 .49975-.14748 .00004 .00553
10.00 2.0015.70  .86-8.0080.25 8.25        * .33333 .00000 .36240-.00753-.00003-1.0318 .00000 .49340-.14032 .00004 .00548
10.00 2.0015.80  .86-8.0080.25 8.25        * .33333 .00000 .35892-.00747-.00003-1.0252 .00000 .48718-.14124 .00004 .00545
10.00 2.0015.90  .86-8.0080.25 8.25        * .33333 .00000 .35551-.00741-.00003-1.0188 .00000 .48107-.13823 .00004 .00544
10.00 2.0016.00  .86-8.0080.25 8.25        * .33333 .00000 .35216-.00735-.00003-1.0124 .00000 .47507-.13530 .00004 .00546
10.00 2.0016.10  .86-8.0080.25 8.25        * .33333 .00000 .34886-.00729-.00003-1.0061 .00000 .46919-.13243 .00004 .00549
10.00 2.0016.20  .86-8.0080.25 8.25        * .33333 .00000 .34563-.00723-.00003-.99996 .00000 .46341-.12964 .00004 .00555
10.00 2.0016.30  .86-8.0080.25 8.25        * .33333 .00000 .34246-.00717-.00003-.99382 .00000 .45775-.12691 .00004 .00562
10.00 2.0016.40  .86-8.0080.25 8.25        * .33333 .00000 .33934-.00712-.00003-.98776 .00000 .45218-.12425 .00004 .00572
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10.00 2.0016.90  .86-8.0080.25 8.25        * .33333 .00000 .32454-.00685-.00003-.95854 .00000 .42582-.11185 .00004 .00644
10.00 2.0017.00  .86-8.0080.25 8.25        * .33333 .00000 .32173-.00680-.00003-.95290 .00000 .42083-.10955 .00004 .00663
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 5.00 1.10 5.10  .41-3.0015.82 3.82 77.769 .33333 .00005 .75569-.38882-.00409-1.0233 .00016 .39740-.06937 .00431 .00531
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 5.00 2.0016.60  .86-3.0015.25 3.25        * .33333 .00000 .34260-.00922-.00039-.98755 .00000 .44737-.12075 .00044 .00582
 5.00 2.0016.70  .86-3.0015.25 3.25        * .33333 .00000 .33955-.00915-.00039-.98164 .00000 .44203-.11824 .00043 .00593
 5.00 2.0016.80  .86-3.0015.25 3.25        * .33333 .00000 .33656-.00908-.00039-.97580 .00000 .43678-.11579 .00043 .00605
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5.00 2.0016.90  .86-3.0015.25 3.25     * .33333 .00000 .33363-.00902-.00038-.97002 .00000 .43163-.11339 .00043 .00620
5.00 2.0017.00  .86-3.0015.25 3.25     * .33333 .00000 .33074-.00895-.00038-.96032 .00000 .42656-.11105 .00042 .00636
5.00 2.0017.10  .86-3.0015.25 3.25     * .33333 .00000 .32790-.00888-.00038-.95868 .00000 .42159-.10877 .00042 .00653
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5.00 2.0017.80  .86-3.0015.25 3.25     * .33333 .00000 .30930-.00845-.00036-.92097 .00000 .38908-.09419 .00040 .00814
5.00 2.0017.90  .86-3.0015.25 3.25     * .33333 .00000 .30681-.00839-.00036-.91583 .00000 .38475-.09229 .00040 .00842
5.00 2.0018.00  .86-3.0015.25 3.25     * .33333 .00000 .30437-.00833-.00036-.91074 .00000 .38048-.09044 .00039 .00870
PROGRAM COMPLETED SUCCESSFULLY
CONTROL FORCED TO NEXT PROGRAM
   000 00000       END OF LOG ENTRY                                                                      NOV   02   15 59
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